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On the viscous motion of a small particle in a rotating cylinder

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The dynamics of a non-neutrally buoyant particle moving in a rotating cylinder filled with a Newtonian fluid is examined analytically. The particle is set in motion from the centre of the cylinder due to the acceleration caused by the presence of a gravitational field. The problem is formulated in Cartesian coordinates and a relevant fractional Lagrangian equation is proposed. This equation is solved exactly by recognizing that the eigenfunctions of the problem are Mittag–Leffler functions. Virtual mass, gravity, pressure, and steady and history drag effects at low particle Reynolds numbers are considered and the balance of forces acting on the particle is studied for realistic cases. The presence of lift forces, both steady and unsteady, is taken into account. Results are compared to the exact solution of the Maxey–Riley equation for the same conditions. Substantial differences are found by including lift in the formulation when departing from the infinitesimal particle Reynolds number limit. For particles lighter than the fluid, an asymptotically stable equilibrium position is found to be at a distance from the origin characterized by $X \approx -V_\tau/\Omega$ and $Y/X \approx (C_S/3\pi\sqrt{2})Re_s^{1/2}$, where V_τ is the terminal velocity of the particle, Ω is the positive angular velocity of the cylinder, Re_s is the shear Reynolds number $a^2\Omega/\nu$, and C_S is a constant lift coefficient. To the knowledge of the authors this work is the first to solve the particle Lagrangian equation of motion in its complete form, with or without lift, for a non-uniform flow using an exact method.

1. Introduction

We address the motion of a small spherical non-neutrally buoyant particle in a rotating cylinder that is filled with a Newtonian fluid. The cylinder is assumed to be rotating at a constant rotation rate around its axis on a horizontal plane. A gravitational field perpendicular to the horizontal plane induces the motion of the particle from the centre of the cylinder. This initial condition is chosen for simplicity only, since other more general initial conditions can be treated with the same methods used in this work.

The problem at hand is of great difficulty, and only an approximate method can be used to solve it. We choose to make approximations in the formulation of the Lagrangian equation of motion and then solve the model equation of motion exactly. An alternative method to study this flow is to solve numerically for the time-dependent Navier–Stokes flow field around the particle and adjust the position of the particle according to the resulting force acting on it. This latter option proves to be very expensive from the point of view of computation time because the equilibrium position, when stable, is reached only after a very long time. We will show that this time is of the order of hundreds to thousands of rotation cycles for characteristic parameters of interest.

Our formulation predicts that even at very small rotation rates a remarkable phenomenon occurs due to lift effects exclusively: the equilibrium position of particles lighter than the fluid is always below (assuming the gravitational acceleration to point down) the horizontal plane containing the axis of the cylinder. This result is in direct contrast with the behaviour predicted by the Maxey–Riley equation which does not include lift effects (Maxey & Riley 1983). The exact solution of the Maxey–Riley equation derived in the present paper predicts that a light particle will reach equilibrium above the central plane. This result is particularly important because the Maxey–Riley equation is often used for calculation of particle motion in non-uniform flows at small but finite particle Reynolds numbers $Re_p = aW_o/\nu$, where a is the radius of the particle, W_o the relative particle-to-fluid velocity and ν the fluid viscosity. We show that even at very small Re_p and shear Reynolds number $Re_s = a^2\Omega/\nu$ (where Ω is the angular velocity of the cylinder) lift effects will force a light particle below the central horizontal plane, provided $Re_p \leqslant Re_s$. This result should be noted when trying to extrapolate the use of the Maxey–Riley equation beyond the limits of infinitesimal Re_p and Re_s for which the equation was originally (and correctly) derived. We also show that the flow configuration under study in this paper can be used to determine experimentally the lift coefficient for a particle in a uniform vorticity field.

The importance of this study is however not only the fundamental significance of the result described above. This flow configuration has been used recently in a variety of different applications related to tissue growth engineering. The concept is that tissue can be grown in a ‘reduced gravity’ environment when placed inside a rotating bioreactor that resembles the fundamental geometry of our problem (see e.g. Gao, Ayyaswamy & Ducheyne 1997). The suspension bioreactor consists of a cylinder that rotates round its horizontal axis. Heavier tissue migrates outward continuously, limiting the number of hours for which tissue can be grown (typically no more than 48 hours). Radial migration rates corresponding to an equivalent gravitational acceleration of the order of one hundredth of the normal gravity on Earth can be generated for typical conditions in the case of heavy particles. Lighter tissue can grow indefinitely as long as the reactor can sustain larger particles. As the cells or tissue grow, the rotational speed of the bioreactor must be adjusted in order to prevent them from migrating to the centre of the reactor (where typically a membrane is placed for exchange of nutrients and waste), or adhering to the walls of the cylinder. Understanding the mechanics of the suspended tissue (or any suspended particle) within the rotating cylinder is paramount to the determination of given desired growth conditions.

Another application of the flow considered in this work is the manufacturing of precision latex microspheres through the use of a rotating latex reactor (Roberts, Kornfeld & Fowlis 1991). The works of Roberts *et al.* (1991) and Gao *et al.* (1997) address the motion of particles under the assumption of negligible history and lift effects. Tio *et al.* (1993) studied the dynamics of heavy and light particles in a periodic Stuart vortex. A numerical study was conducted including both the traditional Basset force and a steady lift force, and the results of the numerical computations were compared to analytical results obtained by considering the drag force to be dominant. The authors reported good agreement between the numerical and analytical studies for the range of parameters considered.

The next section of this work addresses fundamental background information for the development of a model Lagrangian equation of motion that includes linear lift effects. A brief review of the development of previous Lagrangian equations and their ranges of applications are discussed. In § 3 we discuss the proposed model equation and its solution, as well as the determination and stability of the equilibrium point from both physical and mathematical perspectives. In § 4 we present results obtained from the exact solutions of the Maxey–Riley equation and the proposed equation of motion for relevant parameters. The main conclusions of this work are summarized in § 5.

2. Lagrangian formulation of particle motion in viscous flows

Stokes (1850) pioneered the study of viscous particle motion by determining the force acting on a small fixed particle that is subjected to a uniform fluid velocity. Stokes formulated three different problems: (*a*) the steady migration of particles, (*b*) the small-amplitude motion of a pendulum, and (*c*) the uniform rotation of a spherical particle around its axis (cf. Stokes 1966). In all three cases, Stokes related the undisturbed flow or far-stream conditions to the force acting on the sphere, thus creating a direct relationship between the resulting force (or torque) acting on a spherical particle and the kinematics of the free-stream flow. Mathematically, Stokes determined the operator Λ_S that relates the force $\boldsymbol{F}$ acting on a particle to the background flow field $\boldsymbol{U}_o$, such that an expression of the form $\boldsymbol{F} = \Lambda_S(\boldsymbol{U}_o)$ is determined.

In the simplest case (*a*), the resulting Stokes drag formula relates the force exerted on the sphere to the constant free-stream velocity $\boldsymbol{U}_o$, the dynamical viscosity of the fluid μ, and the radius a of the sphere in a linear way, given that the particle Reynolds number $Re (= a|\boldsymbol{U}_o|/\nu)$ is maintained much smaller than unity. In fact, the Stokes drag happens to agree well with experiments for values of Re up to 0.5, but this coincidental fact should not be generally extrapolated to unsteady flows (see discussion below). For the case of steady migration, or equivalently, when a fixed particle is subjected to a constant free-stream velocity, Stokes carried on the integration of pressure and viscous forces on the surface of the particle to arrive at a simple constant operator $\Lambda_S = 6\pi\mu a$, which is the appropriate drag coefficient for a sphere at infinitesimal Re. Implicit in the Stokes derivation for the quasi-steady drag force is that the free-stream velocity U_o must be constant. Since there are no accelerations involved, the choice of reference frame fixed on the solid particle or on a fluid particle in the background flow is immaterial. In the gravitationally induced motion of a light particle through a quiescent fluid, the free-stream velocity is constant (zero) but the particle accelerates until it reaches its terminal velocity. In this case, and in many others of practical interest, the quasi-steady formulation of the problem, i.e. the Lagrangian equation for the particle written as a quasi-steady response to the Stokes drag, incurs an error, and the unsteady contribution from the developing profile near the particle needs to be taken into account. This is particularly true for particles with similar or smaller inertia than the surrounding fluid.

Stokes derived the mathematical relation for the steady drag by solving the flow field using a bi-harmonic equation for the stream function and neglecting inertial effects. The justification for neglecting all inertial effects in the case of steady flow past a small sphere is clear if the momentum equation is cast in non-dimensional form as

$$SlRe_p\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}+\frac{\partial \boldsymbol{w}}{\partial t}\right)+Re_p\boldsymbol{w}(\nabla\cdot\boldsymbol{w})=-\nabla p+\nabla^2\boldsymbol{w}, \tag{2.1}$$

where the Reynolds and the Strouhal ($Sl \equiv a/U_o t_c$) numbers appear naturally from the scaling of the Navier–Stokes equation. In equation (2.1) p stands for the dimensionless pressure, which is normalized by the characteristic dynamic pressure of the flow. The symbols $\boldsymbol{v}$ and $\boldsymbol{w}$ refer to the dimensionless velocity of the particle and the dimensionless relative velocity $\boldsymbol{w}=\boldsymbol{v}-\boldsymbol{u}$, where $\boldsymbol{u}$ is the dimensionless fluid velocity. The characteristic time of unsteadiness of the flow is taken as t_c, and the characteristic length as a. For steady flow past a sphere and for small Re_p, all terms on the left-hand side of equation (2.1) are indeed small (the term containing Sl is identically zero in this case). Here, the definitions of Re and Re_p should be noted. When the particle is in a fixed position, the characteristic velocity is the free-stream velocity. However, when the particle is allowed to move in response to the forces acting on it, the important parameter is not Re based on the free-stream velocity but Re_p, a Reynolds number based on the characteristic slip velocity W_o that is representative of the absolute value of $\boldsymbol{V}-\boldsymbol{U}$. Throughout this work we reserve upper-case symbols to indicate dimensional variables (with obvious exceptions, such as Re_p and Sl). We refer to the corresponding dimensionless quantities by lower-case symbols.

The following subsections present a brief review of previous attempts to model the motion of small particles through viscous flow fields. These sections serve as reference for the arguments leading to a Lagrangian equation suitable for modelling the motion of a small particle in a linear solenoidal velocity vector field.

2.1. *Particle motion in uniform flows – Tchen's equation*

Boussinesq (1885) and Basset (1888) independently extended Stokes' derivation to a case where the particle accelerates through the fluid due to a constant gravitational force but still neglecting the convective terms. Boussinesq and Basset accomplished this by considering only the unsteady and viscous terms, which means that only flows characterized by $Re_p \ll SlRe_p$ are modelled by the resulting equation of motion when departing from the infinitesimal Re_p range (cf. equation (2.1)). Oseen (1927) contributed to the previous work of Boussinesq and Basset, concentrating on the extension of the equations to higher Re_p. The particle equation of motion with a constant forcing (the gravity term) is sometimes referred to as the BBO equation, due to the original contributions of Boussinesq, Basset and Oseen. The BBO equation is an integro-differential equation that has a removable singularity in the integrand of the history term. The history term can be derived directly from the Stokes operator Λ_S using Duhamel's Superposition Theorem (Coimbra & Rangel 2000). The history term is found to be simply $a\Lambda_S \nu^{-1/2}\tilde{\mathscr{D}}^{1/2}(\boldsymbol{V})$, where $\tilde{\mathscr{D}}^{1/2}(\boldsymbol{V})$ represents the half-derivative of the particle velocity (Coimbra 1998), and the generalized differential operator is defined as

$$\left.\begin{aligned}\tilde{\mathscr{D}}^q\boldsymbol{V}&=\frac{1}{\Gamma(-q)}\int_{-\infty}^{t}(t-\sigma)^{-q-1}\boldsymbol{V}(\sigma)\,\mathrm{d}\sigma, && q<0,\\ \tilde{\mathscr{D}}^q\boldsymbol{V}&=\frac{1}{\Gamma(p-q)}\frac{\mathrm{d}^p}{\mathrm{d}t^p}\int_{-\infty}^{t}(t-\sigma)^{p-q-1}\boldsymbol{V}(\sigma)\,\mathrm{d}\sigma, && q\geqslant 0,\end{aligned}\right\} \tag{2.2}$$

where $\Gamma(s)$ is the Gamma (generalized factorial) function of s, and $p-1\leqslant q<p$ with $p=1,2,3$.

An interesting issue is raised when different unsteady flow fields are considered, namely what is the appropriate definition of the characteristic time for the unsteadiness of the free-stream flow? Lovalenti & Brady (1993) argued that for a particle that accelerates from rest in a stationary fluid, the appropriate time scale is the time that it takes for vorticity to be diffused (still under the assumption of small Re_p) over a length scale comparable to its radius. In this case, t_c is of order a^2/ν. Also for this particular situation, neglecting the convective terms for finite but small Re_p is warranted since Sl is of order Re_p^{-1} at initial times. This time scale is not necessarily appropriate when vorticity can be convected by the wake of the particle at higher Re_p.

When a particle accelerates from rest in a quiescent fluid, the value of Sl is initially high due to the high initial acceleration but as the particle approaches terminal velocity this time scale of variation is greatly reduced. The value of Sl thus varies from a large value to zero as the particles reaches terminal velocity. Therefore, the consideration of the unsteady term alone yields incorrect results if the convective terms are neglected for long times. This is because the term containing $SlRe_p$ is not dominant over the convective terms for long times.

Tchen (1947) dealt with the problem of modifying the BBO equation for the case of a uniform but time-dependent free-stream flow field (the free-stream flow field is also called *background* or *undisturbed* flow field in this work). The resulting equation, valid for the limit of infinitesimal Re_p and for uniform unsteady background flows, relates the transient acceleration of the particle to the time-dependent free-stream or background flow velocity. This equation is valid for solid particles or very small bubbles that present no surface motion. In terms of the dimensionless relative velocity $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$, where $\boldsymbol{v}$ and $\boldsymbol{u}$ are the particle and the fluid velocity respectively, Tchen's first equation of motion is written as

$$\tilde{\mathscr{D}}(\boldsymbol{w}) + 3\hbar^{1/2}\tilde{\mathscr{D}}^{1/2}(\boldsymbol{w}) + \boldsymbol{w} = -(1-\alpha)\tilde{\mathscr{D}}(\boldsymbol{u}) + \boldsymbol{g}, \tag{2.3}$$

where α is the fluid-to-particle density ratio and $\hbar \equiv \alpha/(2+\alpha)$. Velocities are made dimensionless by the flow characteristic velocity U_o. Time is made dimensionless by defining a particle characteristic time τ_p given by $a^2/9\nu\hbar$. The vector $\boldsymbol{g}$ is the dimensionless gravity term $(1-\alpha)\tau_p\gamma\boldsymbol{G}/U_o$, where $\boldsymbol{G}$ is the local gravitational acceleration, a is the radius of the spherical particle, ν is the kinematic viscosity of the fluid, and $\gamma \equiv 2/(2+\alpha)$. If dynamic equilibrium between the particle and the fluid is reached, the characteristic velocity U_o can be taken as the terminal velocity $\boldsymbol{V}_\tau$ under the gravitational field $\boldsymbol{G}$. The normalized dimensionless gravity vector in such cases is just $[0\ \ 1]^T$. In operator form, Tchen's equation of motion is simply

$$\lambda(\boldsymbol{w}) = -(1-\alpha)\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{g}, \tag{2.4}$$

where λ is the dimensionless operator

$$\lambda(\boldsymbol{w}) = \tilde{\mathscr{D}} + 3\hbar^{1/2}\tilde{\mathscr{D}}^{1/2} + \lambda_S. \tag{2.5}$$

In dimensionless form, the history term is just $3\hbar^{1/2}\lambda_S\tilde{\mathscr{D}}^{1/2}(\boldsymbol{w})$ and $\lambda_S = 1$. Since Tchen derived equation (2.3) by neglecting the convective terms in equation (2.1), the validity of the equation is restricted to values of $Re_p \ll SlRe_p$, as discussed above. For the case of an oscillating flow of frequency Ω past a fixed particle equation (2.3) is only valid when $aU_o/\nu \ll a^2\Omega/\nu$, or when $a\Omega \gg U_o$. In particular, Tchen's equation represents an accurate model of the particle motion when $aU_o/\nu \ll a^2\Omega/\nu \leqslant 1$, a result that is relevant for the problem of particle motion in a rotating cylinder under consideration in this work. Similarly, a free floating particle in an oscillatory flow of frequency Ω sees the flow with a characteristic time of order Ω^{-1}. One concludes that for small values of Ω it is incorrect to neglect the convective terms when compared to the unsteady term for finite Re_p. Mei & Adrian (1992) studied the correction of the kernel of the history (integral) term for conditions beyond the scope of Tchen's equation, including cases where $Re > 1$ for a fixed particle subject to a harmonically perturbative field. Kim, Elghobashi & Sirignano (1998) corrected Mei's kernel for a freely moving particle and for $\alpha < 0.2$. Coimbra & Rangel (2001) solved equation (2.3) exactly for a sinusoidal flow field including initial transient effects.

2.2. *The Maxey–Riley equation for non-uniform flows*

When the particle is allowed to move with respect to an inertial reference frame in a non-uniform flow field, the formulation of the problem is best done by considering a reference frame moving with the centre of the particle designated as $\boldsymbol{X}(T)$. The inertial reference frame is $\boldsymbol{\chi}$ and a differential reference frame is $\boldsymbol{Z} = \boldsymbol{X} - \boldsymbol{\chi}$. Therefore the relative velocity of the fluid with respect to the moving reference frame $\boldsymbol{X}$ is given by $\boldsymbol{W}(\boldsymbol{Z}, T) = \boldsymbol{V}(T) - \boldsymbol{U}^1(\boldsymbol{\chi}, T)$, where $\boldsymbol{U}^1(\boldsymbol{\chi}, T)$ is the disturbance flow field caused by the presence of the particle as opposed to the undisturbed flow field $\boldsymbol{U}^0(\boldsymbol{\chi}, T)$ that exists in the absence of the particle. The continuity and momentum equations are satisfied for the relative velocity $\boldsymbol{W}(\boldsymbol{Z}, T)$. Maxey & Riley (1983) derived an equation of motion for a small particle moving in a non-uniform flow field by treating differently the inner field (the disturbance field in the vicinity of the particle) and the outer field (the undisturbed field that would exist in the absence of the particle). Their derivation invokes two separate velocity fields $\boldsymbol{W}^0$ and $\boldsymbol{W}^1$ corresponding to the undisturbed and the disturbance flows respectively. The total relative velocity is a simple composition of both contributions such that $\boldsymbol{W} = \boldsymbol{W}^0 + \boldsymbol{W}^1$.

By equating the terms of the undisturbed field in one equation

$$\Delta^{NS}(\boldsymbol{W}^0) = \left(\boldsymbol{G} - \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}T}\right), \tag{2.6}$$

where Δ^{NS} is the Navier–Stokes operator

$$\Delta^{NS}(\boldsymbol{W}^k) = \frac{\partial \boldsymbol{W}^k}{\partial T} + (\boldsymbol{W}^k \cdot \nabla)\boldsymbol{W}^k + \frac{\nabla \boldsymbol{P}^k}{\rho} - \nu \nabla^2 \boldsymbol{W}^k, \tag{2.7}$$

and the cross-derivative convective terms with the remaining terms of the disturbance field in another,

$$\Delta^{NS}(\boldsymbol{W}^1) + (\boldsymbol{W}^0 \cdot \nabla)\boldsymbol{W}^1 + (\boldsymbol{W}^1 \cdot \nabla)\boldsymbol{W}^0 = 0, \tag{2.8}$$

Maxey & Riley (1983) formulated the contribution of the outer field quite generally (equation (2.6)). The contribution of the inner field is more complicated however, and the resulting force was evaluated by neglecting all convective terms in equation (2.8). This consideration leads to the tight restrictions for the particle Reynolds number $aW_o/\nu \ll 1$ and for the so-called shear Reynolds number $a^2U_o/L\nu \ll 1$, where we recall that W_o and U_o are the characteristic relative and background velocities and L is the characteristic length scale of the background flow. The Maxey–Riley equation is

$$\begin{aligned} m_p\tilde{\mathscr{D}}(\boldsymbol{V}) = m_f\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}T} - \frac{m_f}{2}\left(\tilde{\mathscr{D}}(\boldsymbol{V}) - \frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}T}\right) - \frac{a\Lambda_S}{\nu^{1/2}}\tilde{\mathscr{D}}^{1/2}(\boldsymbol{W}) - \Lambda_S(\boldsymbol{W}) + (m_p - m_f)\boldsymbol{G} \\ + \frac{m_f a^2}{20}\tilde{\mathscr{D}}(\nabla^2\boldsymbol{U}) + \frac{a^4\Lambda_S}{6}\tilde{\mathscr{D}}^{1/2}(\nabla^2\boldsymbol{U}) + \frac{a^2\Lambda_S}{6}\nabla^2\boldsymbol{U}. \end{aligned} \tag{2.9}$$

Note that the form of equation (2.9) is not the original form derived by Maxey & Riley because it contains corrections to the lower limit of integration of the history drag term (Reeks & McKee 1984) and to the form of the virtual mass force term (see e.g. Drew & Lahey 1987). It is important to emphasize that the now generally accepted form of the virtual mass force cannot be derived using the method outlined by Maxey & Riley where all convective terms are neglected in the inner field.

The last line in equation (2.9) represents the collective effects of the Faxén corrections for the non-uniformity of the flow. For infinitesimal Re_p the non-uniformity of the flow is felt only through the substantial derivative $\mathrm{D}/\mathrm{D}T$ following a fluid particle and by the second-order Faxén contributions since the flow around the particle is quasi-symmetrical. Maxey & Riley also noted that in the limit of infinitesimal Re_p there should be no effect of particle rotation on the forces acting on the particle due to the same symmetry condition. This is of course not the case if Re_p is small but finite. In fact, for the rotating cylinder case, the velocity gradient and t_c^{-1} are both $O(|\Omega|)$. On one hand, the restrictions imposed by neglecting the convective terms in the derivation of the inner field contribution require that $SlRe_p$ be much larger than Re_p, so that $a^2|\Omega|/\nu \gg Re_p$. On the other hand, the shear Reynolds number Re_s is equal to $a^2|\Omega|/\nu$. Thus, for the rotating cylinder problem $Re_s = SlRe_p$, and therefore the Maxey–Riley equation would only be applicable in this case when $Re_p \ll Re_s \ll 1$. Also the Faxén corrections are zero for the rotating cylinder problem since the Laplacian of the background flow is identically null. Therefore if we wish to study the motion of particles for cases where Re_p is small but non-zero, and for frequencies of practical interest, the Maxey–Riley equation is not adequate. In order to relax the restriction of $Re_p \ll Re_s$, we must include at least a linearized (half-order) correction to convective effects in a uniform-vorticity flow.

2.3. *Inertial lift effects at low Reynolds numbers*

Saffman (1965) studied the lift force on a particle that is drifting with uniform velocity through a linear shear flow under the condition $Re_p \ll Re_s^{1/2}$. This condition should suffice for the application we have at hand, since it allows calculation of flows where Re_p is not restricted to be much smaller than Re_s, allowing the formulation of problems where Re_p is not necessarily infinitesimal but instead satisfies $Re_p \leqslant Re_s$. McLaughlin (1991) has since relaxed the restricion on Re_s and found that the lift force decreases rapidly for $Re_s \geqslant 1$. For our calculation purposes, McLaughlin's extension is not critical because we are interested in flows where $Re_s \sim Re_p \ll 1$. Saffman (1965) linearized the cross-derivative terms in equation (2.8) to arrive at an expression for the lift force $\boldsymbol{F}_L = \Lambda_L(\boldsymbol{W})$, where the constant operator Λ_L is equal to $-C_S\mu a Re_s^{1/2}$, and C_S was numerically determined to be 6.46. The Saffman lift force can be written in vectorial form as

$$F_L = C_S a^2 \left(\frac{\mu\rho_f}{|\boldsymbol{Z}|}\right)^{1/2} \boldsymbol{Z} \times \boldsymbol{W}. \tag{2.10}$$

The total force on a particle that is migrating with uniform velocity in a constant-vorticity ($\boldsymbol{Z}$) flow is

$$F_{S+L} = (\Lambda_s + \Lambda_L)(\boldsymbol{W}), \tag{2.11}$$

where $\Lambda_S = 6\pi\mu a$ and $\Lambda_L = C_S a^2(\mu\rho_f/|\boldsymbol{Z}|)^{1/2}\boldsymbol{Z}\times$. Saffman (1965) also showed that the Saffman lift force relates to the migration velocity $\boldsymbol{V}_\perp$ as $F_L = \Lambda_S(\boldsymbol{V}_\perp)$. Saffman realized that the first inertial effect is felt on the far field. By studying the point source effect of the particle on the far field, Saffman derived the correction to the total force acting on the particle that now carries his name. Saffman's analysis was carried out for unbounded steady flow of constant vorticity. Recently, Asmolov & McLaughlin (1999) used a perturbation method to study the lift effect on a sphere in an unsteady shear flow.

In order to generalize Saffman's result to unsteady flows we realized that both operators Λ_s and Λ_L are linear in $\boldsymbol{W}$ and present the same diffusion coefficient in their respective directions. A point-force approximation indicates that the unsteady formulation of the problem leads to the same form of the history term as found for the drag force alone, i.e. the total unsteady force contribution is a combination of the already familiar unsteady drag and an unsteady lift force given by $a\Lambda_L \nu^{-1/2}\tilde{\mathscr{D}}^{1/2}(\boldsymbol{W})$. This result is mathematically equivalent to the argument used by Basset (1888) to derive the form of the history drag: if $\xi^*(t)$ is the point-source force resulting from the surface integration of stresses caused by flow characterized by the linear partial differential equation $\partial^2\xi/\partial x^2 = \partial\xi/\partial t$ for $\partial/\partial t \to 0$ and $\xi(0) = 0$, then $\tilde{\mathscr{D}}^{1/2}(\xi^*)$ is the unsteady point-source force resulting from the linear differential equation for $\partial/\partial t \nrightarrow 0$.

The form of the unsteady lift force is rigorously valid only if the small contribution of the lift force does not substantially alter the symmetry of the flow around the particle, as the required low values of Re_s and Re_p, and the linearization of the convective effects indicate. Note that mathematically an equivalent symmetric condition for linear flow over a sphere only exists in the limit of infinitesimal Re_p. However, a *de facto* symmetry condition remains up to $Re_p \sim 0.5$. The use of equation (2.11) is relevant only for $Re_p \ll Re_s^{1/2}$ and for both Re_s and Re_p much smaller than unity. The form of the history lift force is also strictly valid only for solenoidal flow fields $\boldsymbol{U}(\boldsymbol{X}, T)$ that satisfy $\nabla \times \boldsymbol{U}(\boldsymbol{X}, T) = \boldsymbol{Z} = \text{constant}$.

3. An equation of motion with lift for uniform-vorticity flows

A consistent equation of motion for solenoidal flows where $\nabla^2\boldsymbol{U}(\boldsymbol{X}, T) = 0$ and $\nabla \times \boldsymbol{U}(\boldsymbol{X}, T) = \boldsymbol{Z} = \text{constant}$ is thus

$$m_p\tilde{\mathscr{D}}(\boldsymbol{V}) = m_f\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}T} - \frac{m_f}{2}\left(\tilde{\mathscr{D}}(\boldsymbol{V}) - \frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}T}\right) - \frac{a\Lambda_S}{\nu^{1/2}}\tilde{\mathscr{D}}^{1/2}(\boldsymbol{W}) - \Lambda_S(\boldsymbol{W})$$
$$+C_S a^2\left(\frac{\mu\rho_f}{|\boldsymbol{Z}|}\right)^{1/2}\boldsymbol{Z}\times\boldsymbol{W} + C_S a^2\left(\frac{\mu\rho_f}{|\boldsymbol{Z}|}\right)^{1/2}\tilde{\mathscr{D}}^{1/2}(\boldsymbol{Z}\times\boldsymbol{W}) + (m_p - m_f)\boldsymbol{G}. \quad (3.1)$$

Simpler forms of equation (3.1) have been proposed in the recent past (see e.g. Ferry & Balachandar 2001). In dimensionless form, the equation of motion is

$$\tilde{\mathscr{D}}(\boldsymbol{v}) = 3\hbar\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}T} - \boldsymbol{w} - 3\hbar^{1/2}\tilde{\mathscr{D}}^{1/2}(\boldsymbol{w}) + C_L(\boldsymbol{\zeta}\times\boldsymbol{w}) + 3C_L\hbar^{1/2}\tilde{\mathscr{D}}^{1/2}(\boldsymbol{\zeta}\times\boldsymbol{w}) + \boldsymbol{g}, \quad (3.2)$$

where $\boldsymbol{\zeta}$ is the dimensionless vorticity, and the lift coefficient C_L is defined as

$$C_L \equiv \frac{C_S}{2\pi}\left(\frac{\hbar}{|\zeta|}\right)^{1/2}. \quad (3.3)$$

The dimensionless vorticity $\boldsymbol{\zeta}$ for the rotating cylinder case equals twice the dimensionless angular velocity $\boldsymbol{\omega}$. Note that we opted for not considering any correction to the virtual mass term since the unsteady part of the virtual mass contribution is only relevant for high-frequency flows and the above equation is only valid for $Re_s \ll 1$. On the other hand, the substantial derivative of the background flow in the virtual mass term presents steady terms associated with the pressure gradient in the rotating cylinder that are very important for the calculation of the particle trajectories. The choice of considering history lift effects with no modification to the virtual mass term can be assessed when the magnitudes of the forces in question are compared to the magnitude of the Stokes drag. The ratio of Saffman lift to Stokes drag is $O(Re_s^{1/2})$. Both ratios of history lift and virtual mass to Stokes drag are of $O(Re_s)$. Any half-order correction to the virtual mass term would be of $O(Re_s^{3/2})$. Since we are concerned with small Re_s, we do not address such corrections.

By decomposing $\boldsymbol{w}$ as $\tilde{\mathscr{D}}(\tilde{\boldsymbol{x}}) - \boldsymbol{u}$, we reduce equation (3.2) to

$$\tilde{\mathscr{D}}^2(\tilde{\boldsymbol{x}}) + 3\hbar^{1/2}\tilde{\mathscr{D}}^{3/2}(\tilde{\boldsymbol{x}} - C_L\boldsymbol{\zeta} \times \tilde{\boldsymbol{x}}) + \tilde{\mathscr{D}}(\tilde{\boldsymbol{x}} - C_L\boldsymbol{\zeta} \times \tilde{\boldsymbol{x}}) = 3\hbar\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 3\hbar^{1/2}\tilde{\mathscr{D}}^{1/2}(\boldsymbol{u} - C_L\boldsymbol{\zeta} \times \boldsymbol{u}) + \boldsymbol{u} - C_L\boldsymbol{\zeta} \times \boldsymbol{u} + \boldsymbol{g}, \tag{3.4}$$

where $\tilde{\boldsymbol{x}}$ is the position vector normalized by $U_o\tau_p$ and U_o is a suitable characteristic velocity of the flow (see discussion below for a precise definition of U_o).

Identities of interest for the rotating cylinder problem are

$$\boldsymbol{v} = \tilde{\mathscr{D}}(\tilde{\boldsymbol{x}}), \tag{3.5}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \tilde{\mathscr{D}}^2(\tilde{\boldsymbol{x}}), \tag{3.6}$$

$$\boldsymbol{u} = \omega\mathbb{J}\tilde{\boldsymbol{x}}, \tag{3.7}$$

$$\boldsymbol{\zeta} \times \boldsymbol{x} = 2\omega\mathbb{J}\tilde{\boldsymbol{x}}, \tag{3.8}$$

$$\boldsymbol{\zeta} \times \boldsymbol{u} = -2\omega^2\tilde{\boldsymbol{x}}, \tag{3.9}$$

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\omega^2\boldsymbol{x}, \tag{3.10}$$

where the 2×2 identity and rotation matrices $\mathbb{I}$ and $\mathbb{J}$ are respectively

$$\mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbb{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.11}$$

The fractional particle equation for the rotating cylinder problem is then

$$\tilde{\mathscr{D}}^2(\tilde{\boldsymbol{x}}) + 3\hbar^{1/2}(\mathbb{I} - 2\omega C_L\mathbb{J})\tilde{\mathscr{D}}^{3/2}(\tilde{\boldsymbol{x}}) + (\mathbb{I} - 2\omega C_L\mathbb{J})\tilde{\mathscr{D}}(\tilde{\boldsymbol{x}}) - 3\hbar^{1/2}\omega(\mathbb{J} + 2C_L\omega\mathbb{I})\tilde{\mathscr{D}}^{1/2}(\tilde{\boldsymbol{x}}) - \omega[(2C_L - 3\hbar)\omega\mathbb{I} + \mathbb{J}]\tilde{\boldsymbol{x}} = \boldsymbol{g}. \tag{3.12}$$

3.1. *The equilibrium point*

From equation (3.12) follows that an equilibrium point exists at values of $\boldsymbol{x}$ that satisfy

$$-\omega[(2C_L - 3\hbar)\omega\mathbb{I} + \mathbb{J}]\tilde{\boldsymbol{x}}_{eq} = \boldsymbol{g}, \tag{3.13}$$

or, after defining U_o as the terminal velocity of the particle,

$$\boldsymbol{x}_{eq} = \left[\frac{-Re_p}{Re_s(1 + Re_s^2(2C_L - 3\hbar)^2/81\hbar^2)} \quad \frac{Re_p(3\hbar - 2C_L)}{9\hbar(1 + Re_s^2(2C_L - 3\hbar)^2/81\hbar^2)}\right]^T, \tag{3.14}$$

where $\boldsymbol{x}_{eq}$ is the equilibrium position vector normalized by the radius of the particle a (in the discussion that follows all dimensionless position vectors are normalized by the radius of the particle and therefore do not carry the tilde used to indicate normalization by $U_o\tau_p$).

For a non-trivial solution of equation (3.14) to exist there should be a non-zero terminal $Re_p = aV_\tau/\nu$, where V_τ is the terminal velocity of the particle moving through a fluid of viscosity ν under the influence of the gravitational field $\boldsymbol{G}$, $\boldsymbol{V}_\tau = (1-\alpha)\tau_p\gamma\boldsymbol{G}$. Note that Re_p carries the sign of the terminal velocity and therefore is allowed to be a negative number. Since we are concerned with cases where $|Re_p| \ll |Re_s|^{1/2} < 1$, the denominator of the x-component of $\boldsymbol{x}_{eq}$ is approximately Re_s so that $x_{eq} \approx -Re_p/Re_s$, where Re_s is defined as $a^2\Omega/\nu = 9\hbar\omega$ and therefore is allowed to be negative. The condition $|Re_p| \leqslant |Re_s|$ is always satisfied as long as there is an equilibrium point such that $|\boldsymbol{x}_{eq}| \leqslant 1$. This condition is possible for $\alpha \geqslant 1$ (see discussion below). The ratio $y_{eq}/x_{eq} = (2C_L - 3\hbar)Re_s/9\hbar$ shows that even though the x-component of the equilibrium point is not strongly affected by the consideration of the lift force, the y-component is because the term $2C_L$ (due to lift) is dominant over the term $3\hbar$. In fact, for a buoyant particle ($\alpha > 1$), the inclusion of the lift force shows that the equilibrium position is at a negative value of the y-coordinate as measured from the centre of rotation. The Maxey–Riley equation predicts a positive value for $y_{eq,MR} \approx Re_p/3$ for $\alpha > 1$ since it does not include lift effects. Inclusion of lift effects gives $x_{eq} \approx -Re_p/Re_s$ and $y_{eq}/x_{eq} = \epsilon_L|Re_s|^{1/2} - Re_s/3 \approx \epsilon_L|Re_s|^{1/2}$ since $2C_LRe_s/9\hbar = \epsilon_L|Re_s|^{1/2}$ with $\epsilon_L = (C_S/3\pi\sqrt{2})\omega/|\omega| \approx \pm 0.4847C_S$. This flow configuration thus presents an interesting way of determining experimentally the Saffman lift coefficient C_S.

We write equation (3.12) as

$$\tilde{\mathscr{D}}^2(\boldsymbol{x}) + 3\hbar^{1/2}(\mathbb{I} - \epsilon_L|Re_s|^{1/2}\mathbb{J})\tilde{\mathscr{D}}^{3/2}(\boldsymbol{x}) + (\mathbb{I} - \epsilon_L|Re_s|^{1/2}\mathbb{J})\tilde{\mathscr{D}}(\boldsymbol{x})$$
$$-\frac{Re_s}{3\hbar^{1/2}} \times (\mathbb{J} + \epsilon_L|Re_s|^{1/2}\mathbb{I})\tilde{\mathscr{D}}^{1/2}(\boldsymbol{x})$$
$$-\frac{Re_s}{9\hbar}\left[\left(\epsilon_L|Re_s|^{1/2} - \frac{Re_s}{3}\right)\mathbb{I} + \mathbb{J}\right]\boldsymbol{x} = \boldsymbol{g} = \left[0 \quad \frac{Re_p}{9\hbar}\right]^T. \quad (3.15)$$

Equation (3.15) has only three parameters, namely Re_s, Re_p and $\hbar$. But Re_p only appears in the non-homogeneous part of the equation, and therefore the solutions are similar around the different equilibrium points given by equation (3.14) for all permissible values of Re_p. As a matter of interest, when the value of ϵ_L is set to zero we recover the complete Maxey–Riley equation for this problem. In the next subsection we present an exact solution procedure for equation (3.15). The solution is then discussed for different values of the parameters Re_p, Re_s and $\hbar$ (or α).

3.2. *Solution of the Lagrangian equation of motion*

In this subsection we present two different but completely equivalent approaches to the solution of the equations of motion.

(*a*) *First method: conjugate operator*

The aim of this method is to reduce the integro- or fractional differential equation (3.15) to an ordinary differential equation. To accomplish that we multiply equation (3.15) by the operator

$$\tilde{\mathscr{D}}^2 + (\mathbb{I} - \epsilon_L|Re_s|^{1/2}\mathbb{J})\tilde{\mathscr{D}} - \frac{Re_s}{9\hbar}\left[\left(\epsilon_L|Re_s|^{1/2} - \frac{Re_s}{3}\right)\mathbb{I} + \mathbb{J}\right]$$
$$-\left[3\hbar^{1/2}\left(\mathbb{I} - \epsilon_L|Re_s|^{1/2}\mathbb{J}\right)\tilde{\mathscr{D}}^{3/2} - \frac{Re_s}{3\hbar^{1/2}}\left(\mathbb{J} + \epsilon_L|Re_s|^{1/2}\mathbb{I}\right)\tilde{\mathscr{D}}^{1/2}\right],$$

and after a straightforward computation obtain

$$\tilde{\mathscr{D}}^4(\boldsymbol{x}) + \mathbb{A}_3\tilde{\mathscr{D}}^3(\boldsymbol{x}) + \mathbb{A}_2\tilde{\mathscr{D}}^2(\boldsymbol{x}) + \mathbb{A}_1\tilde{\mathscr{D}}(\boldsymbol{x}) + \mathbb{A}_0\boldsymbol{x} = \mathbb{B}\boldsymbol{g}, \tag{3.16}$$

where

$$\mathbb{A}_3 = \begin{bmatrix} 2 + 9\hbar(-1 + \epsilon_L^2 Re_s) & 2\epsilon_L(1 - 9\hbar)\sqrt{Re_s} \\ 2\epsilon_L(-1 + 9\hbar)\sqrt{Re_s} & 2 + 9\hbar(-1 + \epsilon_L^2 Re_s) \end{bmatrix},$$

$$\mathbb{A}_2 = \begin{bmatrix} 1 - {\epsilon_L}^2 Re_s + 4\epsilon_L Re_s^{3/2} + \dfrac{2Re_s(-3\epsilon_L\sqrt{Re_s} + Re_s)}{27\hbar} & 2\epsilon_L\sqrt{Re_s} + \dfrac{2(-9 + 1/\hbar)Re_s}{9} + 2\epsilon_L^2 Re_s^2 \\ -2\epsilon_L\sqrt{Re_s} + \left(2 - \frac{2}{9\hbar}\right) Re_s - 2\epsilon_L^2 Re_s^2 & 1 - \epsilon_L^2 Re_s + 4\epsilon_L Re_s^{3/2} + \dfrac{2Re_s(-3\epsilon_L\sqrt{Re_s} + Re_s)}{27\hbar} \end{bmatrix},$$

$$\mathbb{A}_1 = \begin{bmatrix} \dfrac{12\epsilon_L Re_s^{3/2} - 5Re_s^2 + 3\epsilon_L^2 Re_s^3}{27\hbar} & \dfrac{-2Re_s(3 - 3\epsilon_L^2 Re_s + 4\epsilon_L Re_s^{3/2})}{27\hbar} \\ \dfrac{2Re_s(3 - 3\epsilon_L^2 Re_s + 4\epsilon_L Re_s^{3/2})}{27\hbar} & \dfrac{12\epsilon_L Re_s^{3/2} - 5Re_s^2 + 3\epsilon_L^2 Re_s^3}{27\hbar} \end{bmatrix},$$

$$\mathbb{A}_0 = \begin{bmatrix} \dfrac{Re_s^2(-9 + 9{\epsilon_L}^2 Re_s - 6\epsilon_L Re_s^{3/2} + Re_s^2)}{729\hbar^2} & \dfrac{2Re_s^2(-3\epsilon_L\sqrt{Re_s} + Re_s)}{243\hbar^2} \\ \dfrac{6\epsilon_L Re_s^{5/2} - 2Re_s^3}{243\hbar^2} & \dfrac{Re_s^2(-9 + 9{\epsilon_L}^2 Re_s - 6\epsilon_L Re_s^{3/2} + Re_s^2)}{729\hbar^2} \end{bmatrix},$$

and

$$\mathbb{B} = \begin{bmatrix} \dfrac{Re_s(-3\epsilon_L\sqrt{Re_s} + Re_s)}{27\hbar} & \dfrac{Re_s}{9\hbar} \\ \dfrac{-Re_s}{9\hbar} & \dfrac{Re_s(-3\epsilon_L\sqrt{Re_s} + Re_s)}{27\hbar} \end{bmatrix}.$$

We need the initial conditions for the variable $\boldsymbol{x}$ and its derivatives to the third order. The initial position and velocity can be freely chosen as any pair of vectors in $\mathbb{R}^2$. The second derivative is obtained from the equation of motion itself and requires knowledge or assumption of some form of the history of the flow for $t \in]-\infty, 0]$. Finally, the third derivative is obtained from the derivation of the equations of motion. Note that equation (3.16) has the same equilibrium point as the original equation (3.15). This method is an extension of the method suggested by Coimbra & Rangel (1998).

(*b*) *Second method: half-order system of fractional differential equations*

In this second method the usual procedure for reducing an nth-order ordinary differential equation to a system of n first-order ordinary differential equations is transposed to fractional differential equations. To this end we will use the notion of a smooth derivative (*douce* in french) introduced in Matignon & d'Adréa Novel (1995): let f be a causal function – recall that a causal function, or causal signal, is a function with support (the closure of the set where f is non-null) in $\mathbb{R}_+-$ and continuous to the right at $t = 0$, then the smooth derivative d of order α of the function f is defined as

$$\mathsf{d}^\alpha f = \mathscr{D}^\alpha f - Y_{1-\alpha}, \tag{3.17}$$

where $\mathscr{D}^\alpha = Y_{-\alpha}\star$, $\alpha \in \mathbb{R}$ is the derivative of order α in the sense of distributions or generalized functions† and $Y_{-\alpha} \in \mathscr{D}'_+$ is the inverse of $Y_\alpha = t_+^{\alpha-1}/\Gamma(\alpha)$ in the convolution algebra of causal distributions $(\mathscr{D}'_+, \star)$, where $\star$ denotes the convolution product. More details on this notation and terminology may be found in Matignon & d'Adréa Novel (1995) and the references therein.

Equation (3.15) can be cast in the following half-order system of fractional equations:

$$\mathsf{d}^{1/2}\boldsymbol{\xi} = \boldsymbol{A}\boldsymbol{\xi} + \xi_g, \tag{3.18}$$

where

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{x} \\ \mathsf{d}^{1/2}\boldsymbol{x} \\ (\mathsf{d}^{1/2})^2\boldsymbol{x} \\ (\mathsf{d}^{1/2})^3\boldsymbol{x} \end{bmatrix}, \qquad \xi_g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \boldsymbol{g} \end{bmatrix},$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & \mathbb{I} & 0 & 0 \\ 0 & 0 & \mathbb{I} & 0 \\ 0 & 0 & 0 & \mathbb{I} \\ -\boldsymbol{A}_0 & -\boldsymbol{A}_{1/2} & -\boldsymbol{A}_1 & -\boldsymbol{A}_{3/2} \end{bmatrix}, \tag{3.19}$$

which is in block Frobenius form with

$$\begin{aligned} \boldsymbol{A}_0 &= -\frac{Re_s}{9\hbar}\left[\left(\epsilon_L|Re_s|^{1/2} - \frac{Re_s}{3}\right)\mathbb{I} + \mathbb{J}\right], \\ \boldsymbol{A}_{1/2} &= -\frac{Re_s}{3\hbar^{1/2}}(\mathbb{J} + \epsilon_L|Re_s|^{1/2}\mathbb{I}), \\ \boldsymbol{A}_1 &= \mathbb{I} - \epsilon_L|Re_s|^{1/2}\mathbb{J}, \\ \boldsymbol{A}_{3/2} &= 3\hbar^{1/2}(\mathbb{I} - \epsilon_L|Re_s|^{1/2}\mathbb{J}). \end{aligned}$$

The solution of (3.18) is obtained as

$$\boldsymbol{\xi} = E_{1/2}(\boldsymbol{A}, \cdot)\boldsymbol{\xi}_0 + (\mathscr{E}_{1/2}(\boldsymbol{A}, \cdot) \star \xi_g),$$

† Note the distinction between the two derivative operators $\mathscr{D}$ and $\tilde{\mathscr{D}}$. The latter is taken in the classical sense, whereas the former is to be understood in the sense of distributions. For example, consider the Heaviside function H. Its classical derivative $\tilde{\mathscr{D}}H$ is zero everywhere except at the origin. So, the classical derivative is a distribution – the null distribution. However, its generalized derivative $\mathscr{D}H$ is not the null distribution, indeed we have $\mathscr{D}H = \delta$, where δ is the Dirac delta function. The reader not familiar with theory of distributions may consult the classical work of Schwartz (1966).

where $\boldsymbol{\xi}_0$ is the vector with initial conditions

$$\boldsymbol{\xi}_0 = \begin{bmatrix} 0 \\ \dot{\boldsymbol{x}}_0 \\ 0 \\ \boldsymbol{x}_0 \end{bmatrix},$$

and

$$E_{1/p}(\boldsymbol{A}, \hat{t}) = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k \hat{t}_+^{k/p}}{\Gamma(1+k/p)},$$

$$\mathscr{E}_{1/p}(\boldsymbol{A}, \hat{t}) = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k \hat{t}_+^{(k+1-p)/p}}{\Gamma((1+k)/p)},$$

are causal Mittag–Leffler matrices. We use the convention that the hat on $\hat{t}$ indicates that t is a dummy variable.

Remark. (*a*) The prescription of the initial conditions for both methods needs some comments. Indeed, the presence of the history term requires that the solution be known for its entire past from time zero. For example, one may assume that prior to the starting moment, the particle was following the fluid. To accomplish this, some external force should be applied upon the particle, for otherwise the fluid motion would be a solution of the particle equation and, without any external force, this can only occur in the case of a neutrally buoyant particle.

Yet, in many instances we are mostly concerned with the asymptotic motion of the particle as $t \to \infty$. In this case, it is enough to consider impulsive starts for the variable. In other words, we can work within the framework of causal distributions $\mathscr{D}'_+$ and look for a solution of the form $H\boldsymbol{x}$.

For that purpose, consider the 'jump formula' applied to a piecewise smooth function f, that is, f is smooth everywhere except for a finite number of points,

$$\mathscr{D}^p f = \tilde{\mathscr{D}}^p f + \sum_{y \in J_0} \sigma_y^{(0)} \delta_y^{(p-1)} + \cdots + \sum_{y \in J_{p-1}} \sigma_y^{(p-1)} \delta_y, \quad p \in \mathbb{N}, \tag{3.20}$$

where δ is the Dirac delta distribution, $J_i \in \mathbb{N}$ is the (discrete) set where the ith derivative of the function f is discontinuous, $i = 0, \ldots, p-1$, $\delta_y = \tau_y \delta$, $y \in \mathbb{R}$, with $\tau_c : \mathscr{D}' \to \mathscr{D}'$, $c \in \mathbb{R}$, the translation operator for distributions, that is, $\langle \tau_c T, \phi \rangle = \langle T, \tilde{\tau}_{-c} \phi \rangle$, $T \in \mathscr{D}'$, $\phi \in \mathscr{D}$, $\tilde{\tau}_c : \mathscr{D} \to \mathscr{D}$ is the usual translation operator, that is, given $\phi \in \mathscr{D}$, then $(\tilde{\tau}_c \phi)(x) = \phi(x-c)$, $x \in \mathbb{R}$, and

$$\sigma_y^{(k)}(f) = \tilde{\mathscr{D}}^k f(y+0) - \tilde{\mathscr{D}}^k f(y-0), \quad y \in J_k, \ k \in \mathbb{N}$$

denotes the jump of the kth derivative of f at the point y. For the fractional order and for a smooth causal function continuous to the right we consider the identity

$$\mathscr{D}^\alpha f = Y_{1-\alpha} \star \tilde{\mathscr{D}} f + f(0+) Y_{1-\alpha}.$$

Then, within the framework of causal distributions, the initial conditions come into play in a natural way. For example, equation (3.15) for $H\boldsymbol{x}$ can be written as

$$\begin{aligned} &\mathscr{D}^2 H\boldsymbol{x} + \boldsymbol{A}_{3/2} \mathscr{D}^{3/2} H\boldsymbol{x} + \boldsymbol{A}_1 \mathscr{D} H\boldsymbol{x} + \boldsymbol{A}_{1/2} \mathscr{D}^{1/2} H\boldsymbol{x} + \boldsymbol{A}_0 H\boldsymbol{x} \\ &\quad = H\boldsymbol{g} + \left(\mathbb{I}\dot{\delta} - \boldsymbol{A}_{3/2} \frac{1}{2\sqrt{\pi}} \mathsf{pf}(t_+^{-3/2}) + \delta \boldsymbol{A}_1 + \frac{1}{\sqrt{\pi}} t_+^{-1/2} \boldsymbol{A}_{1/2} \right) \boldsymbol{x}_0 \\ &\qquad + \left(\delta \mathbb{I} + \frac{1}{\sqrt{\pi}} t_+^{-1/2} \boldsymbol{A}_{3/2} \right) \dot{\boldsymbol{x}}_0 \end{aligned} \tag{3.21}$$

where pf denotes the finite part, that is, for any test function $\phi \in \mathscr{D}$ we have

$$\langle \mathsf{pf}(t_+^{-3/2}), \phi \rangle = \lim_{\epsilon \to 0} \left\{ \int_\epsilon^\infty t^{-3/2} \phi(t)\, \mathrm{d}t - \frac{2}{\sqrt{\epsilon}} \phi(\epsilon) \right\}.$$

Equation (3.21), which is equivalent to (3.18), clearly shows the convenience of the use of the operator d^α instead of $\mathscr{D}^\alpha$. Also, from regularity considerations we use null initial conditions for the fractional derivative $\mathsf{d}^{1/2}$ and its iterates (compare (3.17)). In this case the solution is C^2.

So for the problem at hand, without loss of generality, we consider the particle to be released from rest, at the centreline. This corresponds to a sudden jump in the external force, from 0 to $\boldsymbol{g}$ at $t = 0$, in other words, the external force field is $\boldsymbol{g}H$; and all initial conditions are null.

The equation for the first method is then

$$\mathscr{D}^4(\boldsymbol{x}) + \mathbb{A}_3 \mathscr{D}^3(\boldsymbol{x}) + \mathbb{A}_2 \mathscr{D}^2(\boldsymbol{x}) + \mathbb{A}_1 \mathscr{D}(\boldsymbol{x}) + \mathbb{A}_0 \boldsymbol{x} = \left(\mathbb{B} + \mathbb{I}\dot{\delta} - \boldsymbol{A}_{3/2} \frac{1}{2\sqrt{\pi}} \mathsf{pf}(t_+^{-3/2}) + \delta \boldsymbol{A}_1 + \frac{1}{\sqrt{\pi}} t_+^{-1/2} \boldsymbol{A}_{1/2} \right) \boldsymbol{g}.$$

(b) To compute $\mathscr{E}_{1/p}(\boldsymbol{A}, t)$ we consider the Jordan canonical form of $\boldsymbol{A}$: $\Lambda = \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$. Then, by Proposition 1 (see Appendix) we obtain $\mathscr{E}_{1/p}(\boldsymbol{A}, t) = \boldsymbol{S}\mathscr{E}_{1/p}(\Lambda, t)\boldsymbol{S}^{-1}$. Thus, if $\boldsymbol{A}$ can be put in diagonal form, we have $\mathscr{E}_{1/p}(\boldsymbol{A}, t) = \boldsymbol{S}\, \mathrm{diag}[\mathscr{E}_{1/p}(\lambda_i, t)]\boldsymbol{S}^{-1}$, where λ_i, $i = 1, \ldots, 8$ are the eigenvalues of $\boldsymbol{A}$ counted with multiplicities. We note that in all computations carried out in this work, the eigenvalues were distinct and consequently $\boldsymbol{A}$ could be put in diagonal form. For the sake of completeness, the general case when the geometric multiplicity may differ from the algebraic multiplicity is explained in the Appendix.

(c) The required convolution $\mathscr{E}_{1/2}(\boldsymbol{A}, \hat{t}) \star \boldsymbol{\xi}_g$ is computed as

$$\mathscr{E}_{1/2}(\lambda, \hat{\tau}) \star H = \int_0^t \mathscr{E}_{1/2}(\lambda, t - \tau)\, \mathrm{d}\tau = \frac{1}{\lambda}(\exp(\lambda^2 t)\, \mathrm{erfc}(-\lambda\sqrt{t}) - 1).$$

Because the coefficient matrix is simpler in the second method, we select it for the analysis that follows.

3.3. *Stability of the equilibrium point*

In this subsection we study mathematically the stability of the equilibrium point. Such analysis is essential to connect the solutions of the model equations to observable experimental data. In a recent note, Druzhinin (2000) raised questions regarding the stability of stationary solutions of Tchen's equation (2.3). If the solutions are not stable, then they will most likely not be observable experimentally. Therefore a rigorous stability analysis of the equilibrium point seems appropriate.

We stress that the discussion below is only valid for $Re_p \leqslant Re_s \ll 1$. This range of validity comes from the derivation of equation (3.15) and can be easily understood from a physical point of view. A particle will stabilize in its equilibrium position where the resultant of all the forces acting on it is zero if the equilibrium of forces is stable (a precise definition of stability is given below). The terminal value of the parameter Re_p must be smaller than or at most equal to Re_s in order to make the equilibrium point a strong attractor for the solution. This condition is clearer if we take the limit of $Re_s \to 0$. If Re_p is larger than Re_s then the particle will diverge from its equilibrium point with terminal velocity V_τ. If $Re_p \leqslant Re_s$ the particle is attracted to the equilibrium point and will eventually reach dynamical equilibrium if the equilibrium point is stable. A light particle has an equilibrium position below the horizontal plane due to lift for $Re_s < 1$ because the two radial forces in this problem, lift and pressure gradient, scale as $Re_s^{1/2}$ and Re_s respectively. Therefore, for $Re_s < 1$, the lift force is the dominant of the two. The steady lift force always points outwards in this flow. The only other force that can balance the resulting outward force is the body force (combined weight–buoyancy) which points upwards for a light particle (note that the Stokes drag is tangential to the velocity field at the equilibrium position). For the body force to have an inward component, the particle must be in either the third or fourth quadrant. In other words, the particle must be below the centreplane. If the lift force is ignored, the pressure gradient force is the only purely radial force and the argument is reversed since the pressure gradient points inwards and therefore the radial component of the body force at equilibrium must point outwards. This would place the equilibrium position of a particle with no lift in the first and second quadrants, above the middle horizontal plane.

An analogous argument can be made for a heavy particle leading to the conclusion that when $Re_s < 1$ the equilibrium point for a heavy particle is above the centreplane when lift is considered and below the centreplane when lift is neglected. However, the equilibrium for a heavy particle is unstable because the only radial force pointing inwards is insufficient to bring the particle to a steady orbit. The magnitude of the pressure gradient is only capable of maintaining a fluid (or neutrally buoyant) particle in a circular orbit. This means that once past the equilibrium point there is no force that can bring the particle back to a smaller radius. The heavy particle thus migrates outwards continuously, as the orbit cannot be arrested. The equilibrium point for a light particle is stable because the radial force that depends on the radial position points to the direction of smaller radius (inwards). Therefore, this force does not grow unbounded as in the case of the heavy particle.

In the literature, there are many notions of stability: Lyapounov and asymptotic stability (for orbits), structural stability (for vector fields or phase portrait), ergodic stability (for the ergodic property) and so on. For the problem at hand, where we want to analyse the local stability of an equilibrium point, we find the notions of Lyapounov and asymptotic stability appropriate.

Accordingly, with respect to the present problem, we say that the equilibrium point

$$\boldsymbol{\xi}_{eq} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \boldsymbol{x}_{eq} \end{bmatrix}$$

in the phase space $\mathbb{R}^8$ is (Lyapounov) *stable* if for any neighbourhood $\mathscr{V} \subset \mathbb{R}^8$ of ξ_{eq} there exists another neighbourhood $\mathscr{U} \subset \mathbb{R}^8$ of ξ_{eq} such that any solution curve starting in $\mathscr{U}$ is defined for all $t > 0$ and will not leave the neighbourhood $\mathscr{V}$. Otherwise, it is said to be *unstable*. It is *asymptotically stable* if it is Lyapounov stable and if there is a neighbourhood $\mathscr{V} \subset \mathbb{R}^8$ of ξ_{eq} such that any solution curve starting in $\mathscr{V}$ is defined for all $t > 0$ and converges towards ξ_{eq}. Due to the linearity of the equations, the basin of an asymptotically stable equilibrium point ξ_{eq} (defined as the set of all solution curves that converges towards ξ_{eq}) is the whole $\mathbb{R}^8$.

In Matignon & d'Adréa Novel (1995) it is shown that for a non-null eigenvalue λ, $\mathscr{E}^j_{1/p}(\lambda, t)$, where the power j is taken with respect to the convolution product, is unstable if $|\arg(\lambda)| < \pi/2p$, is stable if $|\arg(\lambda)| \leqslant \pi/2p$, and is asymptotically stable, although converging only as $\sim \lambda^{-1-j}t^{-1-1/p}$, if $|\arg(\lambda)| > \pi/2p$. Thus, in order to

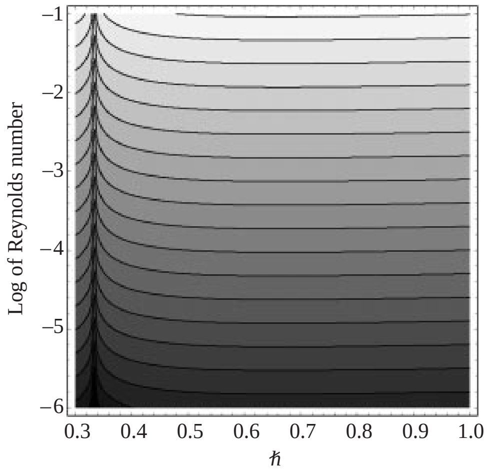

FIGURE 1. Contour plot of the logarithm of the absolute value of $R(\mathrm{Re}(\mathsf{P}_{\pi/4})/16, \mathrm{Im}(\mathsf{P}_{\pi/4})/(-48\sqrt{\hbar}x))$ as a function of $\hbar$ and $\log Re_s$ in the range $(\hbar, Re_s) \in [0.3, 1] \times [10^{-6}, 10^{-1}]$. Equally spaced contours ranges from -80 (darker) to -20 (lighter).

determine the stability characteristics of the equilibrium point for the problem at hand (with $p = 2$) it is only necessary to study the positioning of the spectrum of the matrix $\boldsymbol{A}$ with respect to the stability region $\Lambda_{\pi/4} = \{z \in \mathbb{C} : |\arg(z)| \leqslant \pi/4\}$. We also need to compute the geometric multiplicity, i.e. the number of linearly independent eigenvalues associated with a root, in the case that this root lies on the boundary of the stability region.

Our main findings on the stability of the equilibrium point can be summarized as follows (see the Appendix for a proof).

THEOREM 1. *Under the assumptions leading to the model fractional particle equation* (3.12)*, the following hold:*

(*a*) *for a neutrally buoyant particle* ($\hbar = 1/3$) *the equilibrium point is stable* (*note that for* $\hbar = 1/3$ *the solution is trivial for the chosen initial conditions*);

(*b*) *the equilibrium point for a very heavy particle, that is, when* $\alpha \ll 1$*, is unstable*;

(*c*) *there exists a range of values of* $\hbar$ *close to* $1/3$ *such that the equilibrium point is*

(i) *asymptotically stable for particles lighter than the fluid*;

(ii) *unstable, for particles heavier than the fluid.*

This theorem determines the existence of regions of stability in the parameters space $(\hbar, Re_s)$ or (α, ω). To form an idea of the extent of the latter, we plot the behaviour of some relevant parameters in the region of interest of small shear Reynolds number. Without loss of generality, we consider a cylinder rotating counter-clockwise (the solution for the reverse rotation is just a reflection of the former with respect to the vertical axis).

We know that for a neutrally buoyant particle there are always two eigenvalues on the boundary of $\Lambda_{\pi/4}$ (see the proof of Theorem 1 in the Appendix). Figures 1 and 2 show that, within the framework of low Reynolds number, the converse is also true, that is, if some eigenvalue lies on the boundary of $\Lambda_{\pi/4}$, then the particle is neutrally buoyant.

A necessary condition for the existence of a root on the line $\{Re^{i\pi/4} : R > 0\}$ is the coincidence of the roots of the real and imaginary parts of the polyno-

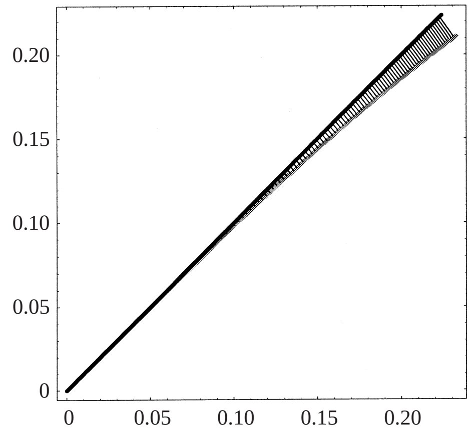

FIGURE 2. $\tilde{\phi}_1$ branch of λ for 100 equally spaced values of ω ($= Re_s/9\hbar$) in the range [0.001, 0.1]. Thick black line: $\{Re^{i\pi/4} : R > 0\}$ and thick grey line: $(\sqrt{2}/2)\sqrt{-1+\sqrt{1+4\,i\omega}}$.

mial $\mathsf{P}_{\pi/4}(x;\hbar,Re_s) = \mathsf{P}(x+ix;\hbar,Re_s)$. To find when it happens, we use the resultant $R(\mathrm{Re}(\mathsf{P}_{\pi/4})/16, \mathrm{Im}(\mathsf{P}_{\pi/4})/(-48\sqrt{\hbar}x))$. In figure 1 is shown the absolute value of the logarithm of the absolute value of the resultant as a function of $\hbar$ and $\log Re_s$ in the range $(\hbar, Re_s) \in [0.3, 1] \times [10^{-6}, 10^{-1}]$. As we can see from figure 1 the only value of $\hbar$ for which there is an eigenvalue on the line $\{Re^{i\pi/4} : R > 0\}$ is $\hbar = 1/3$, that is, for neutrally buoyant particles. Bearing in mind that the spectrum varies continuously with the parameters, we conclude that for the range of parameters considered, the spectrum crosses the boundary of the stability region only at $\hbar = 1/3$. Thus, by Theorem 1, the equilibrium point is asymptotically stable for $(\hbar, Re_s) \in]1/3, 1] \times [10^{-6}, 10^{-1}]$ and unstable for $(\hbar, Re_s) \in [0.3, 1/3[\times [10^{-6}, 10^{-1}]$.

Finally, to cover the region of particles heavier than the fluid we turn again to the variables α and ω. Reckoning that the branch $\tilde{\phi}_1$ (see the Appendix for a definition) is analytic, we use a Cauchy–Kowalevski procedure to determine the coefficients of its Taylor series around $\alpha = 1$. That is, we solve the implicit equation

$$\frac{\partial^n}{\partial\alpha^n}\bar{\mathsf{P}}(\tilde{\phi}_1(\alpha;\omega);\alpha,\omega)|_{\alpha=1} = 0$$

for $\tilde{\phi}_1^{(n)}(1;\omega)$ and compute

$$\tilde{\phi}_1(\alpha;\omega) = \sum_{n=0}^{\infty} \frac{\tilde{\phi}_1^{(n)}(1;\omega)}{n!}(\alpha-1)^n\dagger.$$

In figure 2 we show the values of this branch for 100 equally spaced values of ω in the range $\omega \in [0.001, 0.1]$. Also, for reference we plot the leg $\{Re^{i\pi/4} : R > 0\}$ of the boundary of the stability region $\Lambda_{\pi/4}$ and the curve $\omega \mapsto (\sqrt{2}/2)\sqrt{-1+\sqrt{1+4\,i\omega}}$ representing the unstable eigenvalue in the limit of an infinitely heavy particle ($\alpha = 0$) (see the Appendix). As can be seen from figure 2, the branch $\tilde{\phi}_1$ varies

† In the computations we must truncate the series at some point. For that purpose, we estimate the truncation error at the kth term of the series: $[(\alpha-1)^{k+1}/(k+1)!]\,|\tilde{\phi}_1^{(k+1)}(\theta)|$, $\theta \in [0, 1]$ as $|\tilde{\phi}_1^{(k+1)}(\theta)| \cong |\tilde{\phi}_1^{(k+1)}(1) + \tilde{\phi}_1^{(k+2)}(1)(\theta-1) + \tilde{\phi}_1^{(k+3)}(1)(\theta-1)^2/2| \leqslant |\tilde{\phi}_1^{(k+1)}(1)| + |\tilde{\phi}_1^{(k+2)}(1)| + |\tilde{\phi}_1^{(k+3)}(1)/2|$ and stop when this estimate is smaller than some prescribed level of accuracy. For the eyeball norm of the envisaged graphic representation we use 10^{-3} for this level.

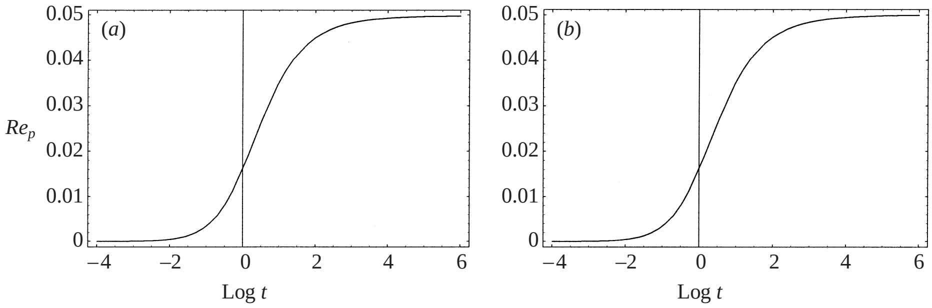

FIGURE 3. Particle Reynolds number Re_p as a function of dimensionless time for $Re_s = Re_p = 0.05$: (a) with lift, (b) without lift.

smoothly from its value $\tilde{\phi}_1(1;\omega) = \sqrt{\omega}e^{i\pi/4}$ for a neutrally buoyant particle to $\tilde{\phi}_1(0;\omega) = (\sqrt{2}/2)\sqrt{-1+\sqrt{1+4i\omega}}$ in the limit of infinitely heavy particles. In other words, the equilibrium point of the motion of any particle heavier than the fluid is unstable for low Reynolds numbers – and so is hardly observable.

4. Results

In order to illustrate the dynamic behaviour of the particles for different flow conditions, we choose characteristic values for Re_s, Re_p and α (or $\hbar$) and discuss deviations from the behaviour of the chosen case based on the mathematical properties of the equations. We choose a base case $|Re_s| = |Re_p| = 0.05$, and values of $\alpha = 1000$, 2 and 0.999. Note that for $\alpha = 0.999$, Re_p is negative. The choice of $|Re_s| = |Re_p| = 0.05$ is motivated by the fact that Re_p can, at most, be equal to Re_s for the equation of motion to be valid, making the equality a limiting case. The values of α were chosen based on limiting cases ($\alpha \gg 1$, $\alpha \approx 1$, and $\alpha \leqslant 1$). The limiting value of $\alpha \to 1^-$ is shown in order to illustrate the unstable equilibrium for particles heavier than the fluid. We note that results for different Re_p are similar, as long as $Re_p \leqslant Re_s$. In particular, the values of the instantaneous particle Reynolds number $\boldsymbol{W}a/\nu$ and the approach to the terminal value of Re_p are virtually identical for the whole spectrum of parameters in the problem, including lift effects or not. This result is particularly important because the approach to equilibrium in the y-direction is strongly dependent on the inclusion of lift, but the approach to terminal particle Reynolds number is not. In fact, the approach to equilibrium in both coordinates is very strongly dependent on the parameters of the problem, as opposed to the approach to terminal Re_p. Figure 3 illustrates the approach to terminal Re_p for the base case including lift effects or not. Deviations from this curve are minimal for all parameters, as long as the instantaneous value of particle Reynolds number is normalized by Re_p. Note that lift effects are not relevant to either the terminal value nor to the approach rate. This result can be easily understood given the fact that the steady drag force is virtually co-linear with the gravitational field for $Re_p \leqslant Re_s$. For the range of parameters in this problem ($Re_p \leqslant Re_s \ll 1$) the asymptotic value of terminal Re_p is reached at $t \sim 10^6\tau_p$ for all cases where the particle reaches terminal Re_p (in other words, for light particles).

Figure 4 shows the effect of lift on the trajectory of a light (hollow) solid particle with fluid-to-particle density ratio $\alpha = 1000$. Given that the presence of surfactants inhibits surface motion on bubbles, the behaviour shown in figure 4 is also representative of surfactant-contaminated bubbles. The trajectories in figure 4 correspond to a total of

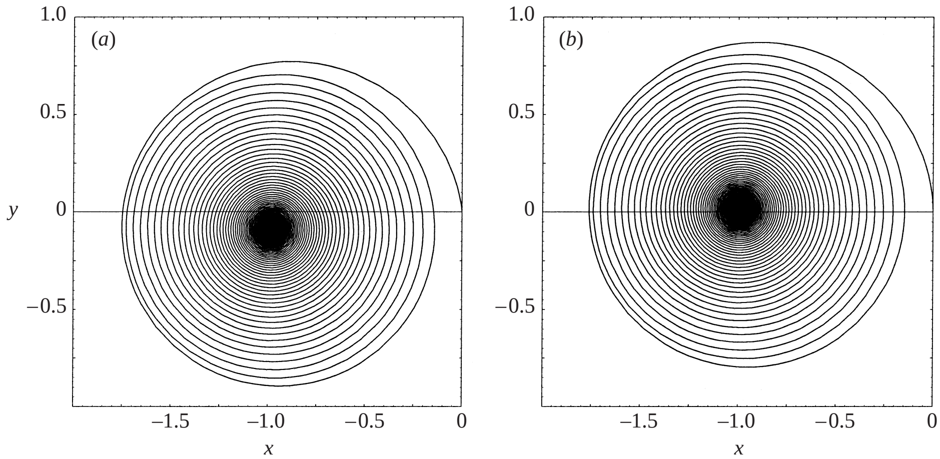

FIGURE 4. Particle trajectories for $\alpha = 1000$ and $Re_s = Re_p = 0.05$, and for 120 rotation cycles of the cylinder: (a) with lift, (b) without lift.

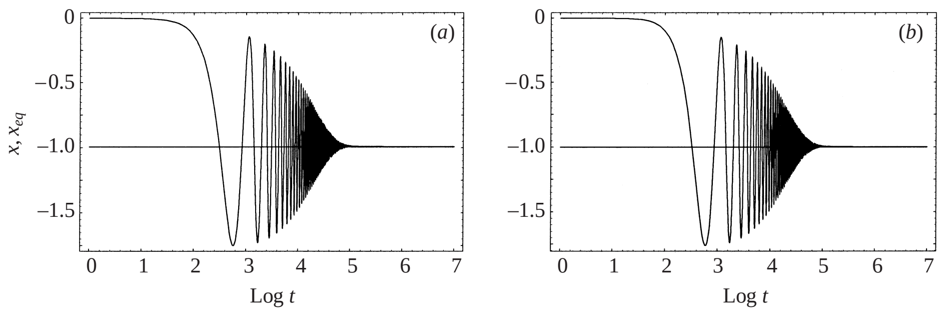

FIGURE 5. Approach to equilibrium in the x-direction for $\alpha = 1000$ and $Re_s = Re_p = 0.05$: (a) with lift, (b) without lift.

120 consecutive periods after the particle is released from the centre of the cylinder. Figure 4 shows that the dynamical behaviour of the particle is essentially not affected by inclusion of lift effects. However the position of the equilibrium point in the y-coordinate is fundamentally different when lift effects are considered (see figures 5 and 6). Decreasing the value of Re_p changes only the position of the equilibrium point. Decreasing both Re_s and Re_p changes the approach time to equilibrium in each coordinate, but not the approach to equilibrium for terminal Re_p (see figures 7, 8 and 9). For example, for $Re_s = Re_p = 0.005$ it takes 100 times longer to reach equilibrium in each coordinate than for $Re_s = Re_p = 0.05$, but in both cases the respective values of terminal Re_p are reached for $t \sim 10^6$, as the linearity of the problem determines. For lower values of both Reynolds numbers, lift effects are reduced as the equilibrium position in the y-coordinate approaches the middle-horizontal plane passing through the origin of the coordinate system. However, the approach is remarkably different when lift effects are included because the particle is always in the opposite quadrant from where it is predicted to be if lift effects are neglected. This result is true for all conditions where the equations are valid and for all density ratios.

Figure 10 shows the trajectory of a particle that is two times lighter than the surrounding fluid ($\alpha = 2$) for $Re_s = Re_p = 0.05$. As the influence of the pressure

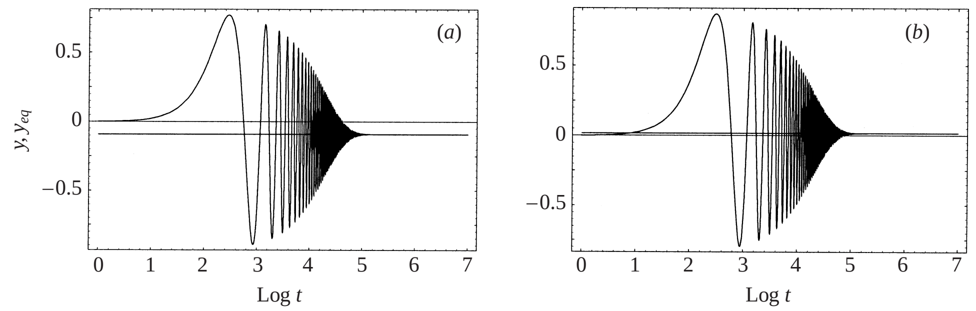

FIGURE 6. Approach to equilibrium in the y-direction for $\alpha = 1000$ and $Re_s = Re_p = 0.05$: (a) with lift, (b) without lift.

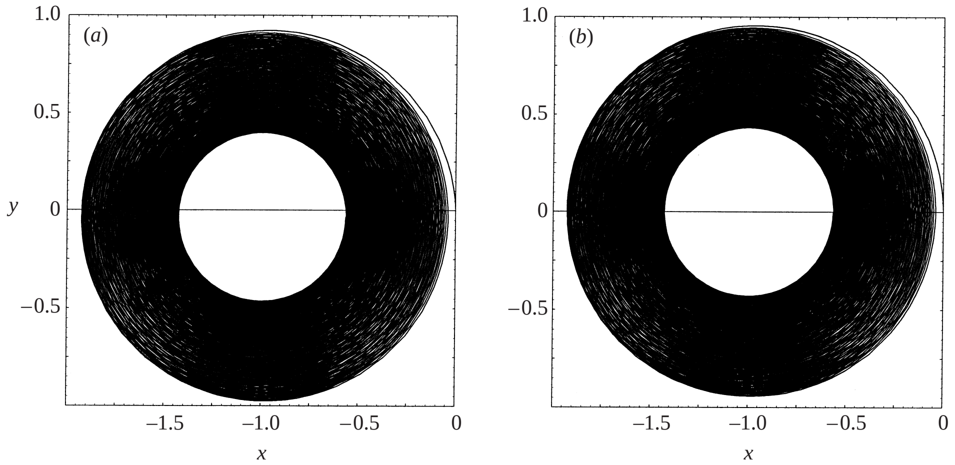

FIGURE 7. Particle trajectories for $\alpha = 1000$ and $Re_s = Re_p = 0.005$, and for 120 rotation cycles of the cylinder: (a) with lift, (b) without lift.

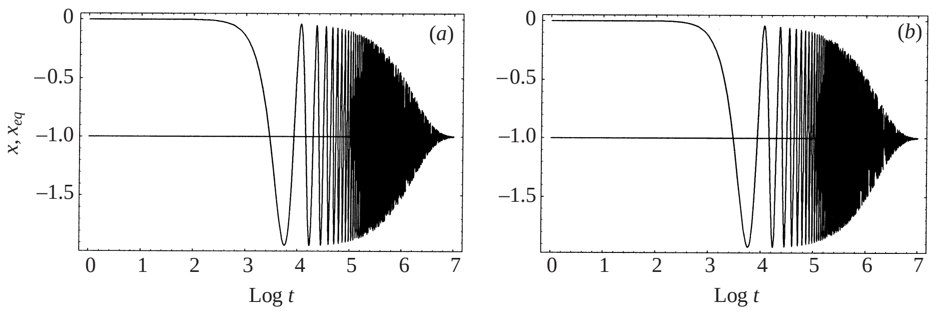

FIGURE 8. Approach to equilibrium in the x-direction for $\alpha = 1000$ and $Re_s = Re_p = 0.005$: (a) with lift, (b) without lift.

gradient is smaller than for the case of $\alpha = 1000$, the particle migrates to its equilibrium position at a slower rate. After 120 cycles the particle has not completely converged to its equilibrium position. As in the other cases, the approach to equilibrium in the x-direction is hardly affected by lift effects (figure 11), but the approach to equilibrium and the equilibrium position itself in the y-direction are strongly affected by inclusion of lift (figure 12).

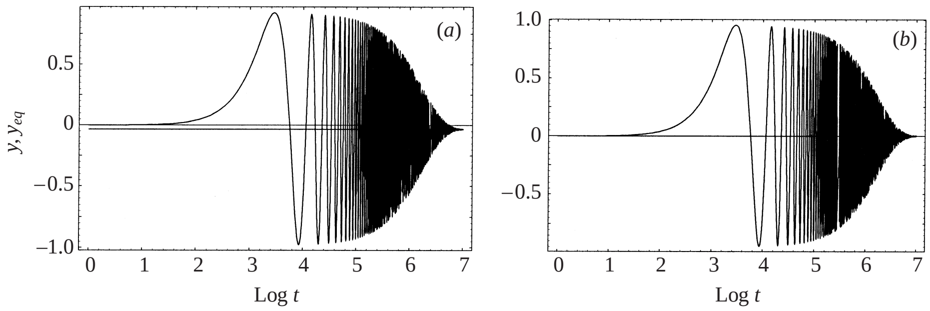

FIGURE 9. Approach to equilibrium in the y-direction for $\alpha = 1000$ and $Re_s = Re_p = 0.005$: (a) with lift, (b) without lift.

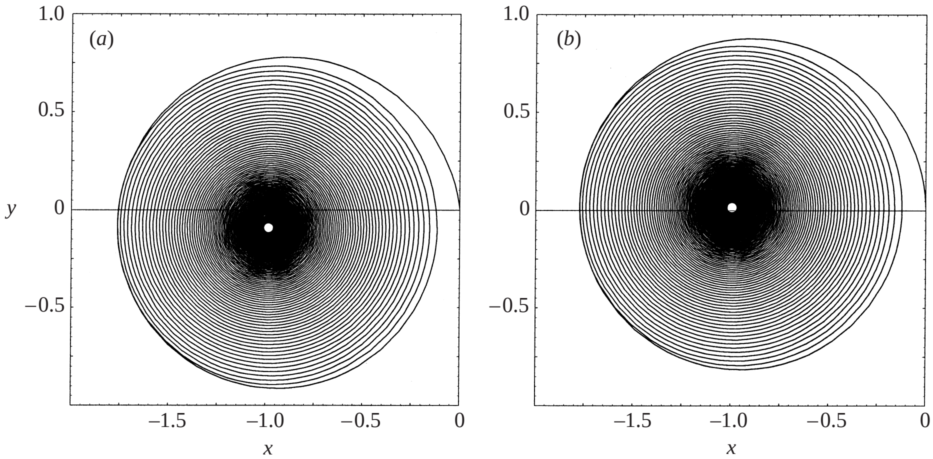

FIGURE 10. Particle trajectories for $\alpha = 2$ and $Re_s = Re_p = 0.05$, and for 120 rotation cycles of the cylinder: (a) with lift, (b) without lift.

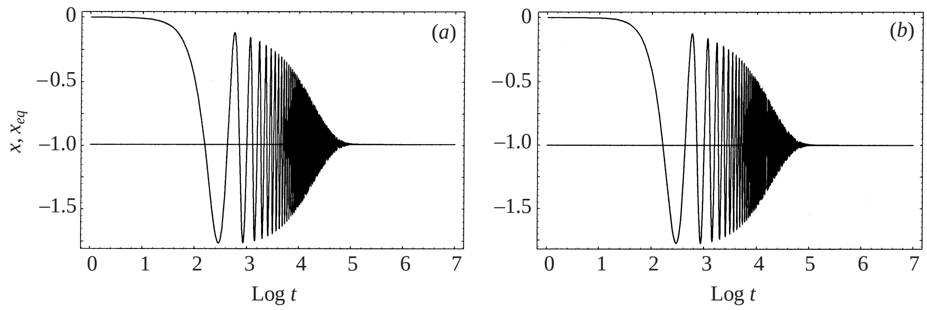

FIGURE 11. Approach to equilibrium in the x-direction for $\alpha = 2$ and $Re_s = Re_p = 0.05$: (a) with lift, (b) without lift.

Figure 13 shows a very different behaviour for a particle that is slightly heavier than the fluid ($\alpha = 0.999$) for $Re_s = -Re_p = 0.05$. Note that in this case the terminal velocity V_τ is negative and therefore Re_p is negative. In this case the particle 'lags' the fluid, and because of that, the equilibrium position in the x-direction is positive for a counter-clockwise rotation of the cylinder. According to the previous discussion in § 3.3, the lift force 'maintains' the heavy particle above the middle plane of

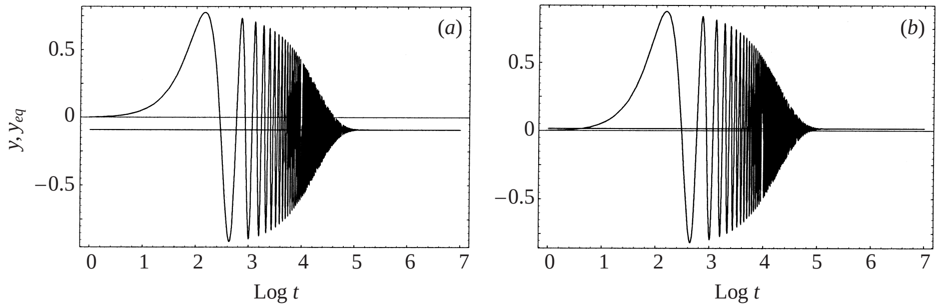

FIGURE 12. Approach to equilibrium in the y-direction for $\alpha = 2$ and $Re_s = Re_p = 0.05$: (a) with lift, (b) without lift.

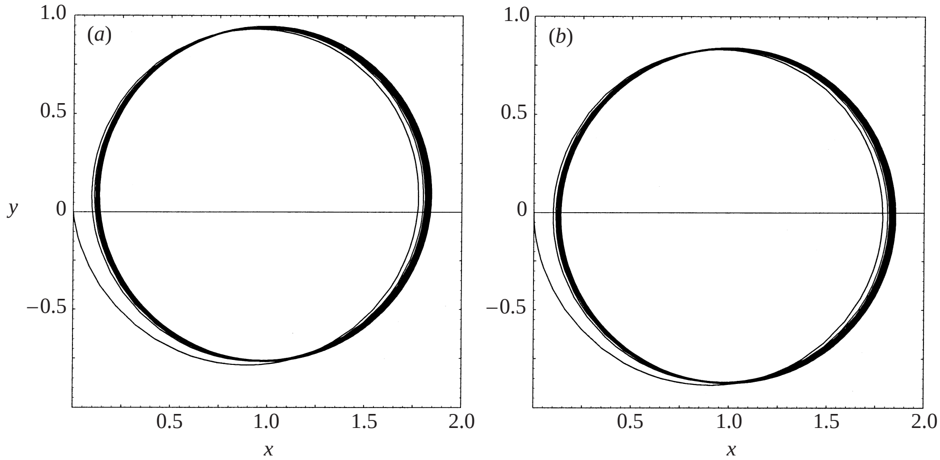

FIGURE 13. Particle trajectories for $\alpha = 0.999$ and $Re_s = -Re_p = 0.05$, and for 120 rotation cycles of the cylinder: (a) with lift, (b) without lift.

the cylinder. The influence of the pressure gradient is very small as $\alpha \to 1-$, and therefore the particle migrates very slowly through the streamlines. After 120 cycles the particle is still very close to its first orbit around the equilibrium point. We proved in § 3.3 that the equilibrium is unstable for this case and the particle does not converge to the equilibrium point. Indeed the particle oscillates around the equilibrium position for a very long time ($t \sim 10^6$) before it starts diverging rapidly (figures 14 and 15). As discussed in § 3.3, once past the equilibrium point a heavy particle cannot decrease the radius of its trajectory because the dominant lift force pointing away from the origin increases with the radial position.

5. Conclusions

This work contributes in many distinct ways to the understanding of viscous particle motion in non-uniform flows:

(1) We provide a comparison between two Lagrangian equations of motion for small particle (Re_p) and shear (Re_s) Reynolds numbers. The two Lagrangian equations differ by whether or not steady and unsteady lift effects are considered. We apply these equations to the motion of a small particle in a rotating cylinder whose axis is in a plane perpendicular to the gravitational acceleration. We show that substantial

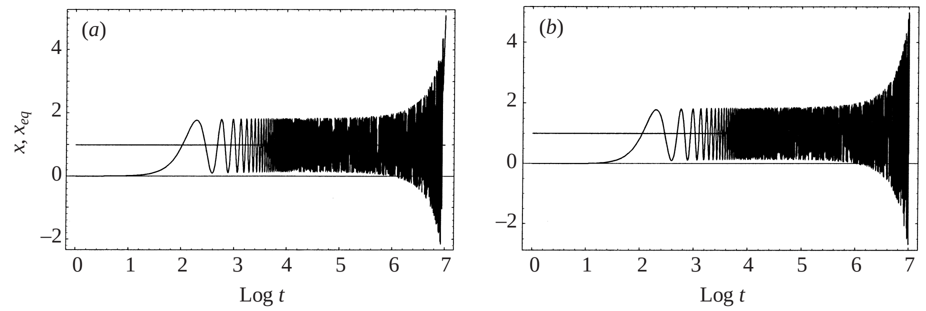

FIGURE 14. Unstable equilibrium in the x-direction for $\alpha = 0.999$ and $Re_s = -Re_p = 0.05$: (a) with lift, (b) without lift.

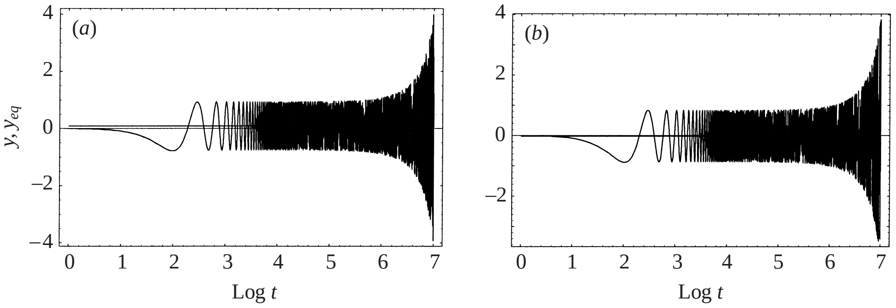

FIGURE 15. Unstable equilibrium in the y-direction for $\alpha = 0.999$ and $Re_s = -Re_p = 0.05$: (a) with lift, (b) without lift.

differences are observed in the motion of the particle in the direction collinear with the gravitational field when lift effects are considered.

(2) The two distinct Lagrangian equations are solved exactly using principles of Fractional Calculus. The methods derived in this work for Lagrangian motion of small particles provide the first full solution to the Maxey–Riley equation (2.9) for a non-uniform flow. We show that it is also possible to solve the proposed equation with linear lift effects (3.1) using two distinct and exact methods.

(3) We show that light particles converge to an equilibrium point that is asymptotically stable and therefore should be observable experimentally.

(4) We discuss the mathematical nature of the equilibrium point, and we prove that a light particle ($\alpha > 1$) always stabilizes in a different quadrant when lift effects are included than the solution given by the Maxey–Riley equation. This result is true for the range of parameters under study ($Re_p \leqslant Re_s \ll 1$) and is explained from physical and mathematical standpoints. The reader should note that this result does not indicate that the Maxey–Riley equation is incorrect. One should recall that the Maxey–Riley equation was derived for the limit of infinitesimal Re_p and Re_s. In the present paper we deal with very small but finite Re_p and Re_s.

(5) We show that the stability of the equilibrium point is not affected by lift effects in the range of parameters under study. We show that the approach to equilibrium in the horizontal direction is not affected by steady or unsteady lift forces. We also show that in the vertical direction (collinear with the gravitational field) lift effects are relevant at small rotations (small Re_s), provided that $Re_p \leqslant Re_s$.

(6) We prove that the equilibrium point for a particle heavier than the fluid is unstable.

(7) We show that the Maxey–Riley equation only approximates the results including lift effects when $Re_p \ll Re_s \ll 1$. This corresponds to the case where the equilibrium point approaches the origin (the origin of the coordinate system is at the axis of rotation of the cylinder). The Maxey–Riley equation predicts an approach to zero for the vertical coordinate of the equilibrium point from 0^+ for a particle lighter than the fluid. Inclusion of lift effects shows that the approach to zero is from 0^-. We therefore conclude that the Maxey–Riley equation, although being correct for the regime $Re_p \to 0$ and $Re_s \to 0$, may yield qualitatively incorrect results for flows characterized by small but finite Re_s and Re_p.

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Appendix

A.1. *Basic properties of the causal Mittag–Leffler matrix*

For completeness, and to aid the computations, we derive some basic properties of the causal Mittag–Leffler matrix.

PROPOSITION 1. *Let* $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathrm{Mat}(\mathbb{C}, n)$.

(*a*) $\mathsf{d}^{1/p} E_{1/p}(\boldsymbol{A}, \hat{t}) = \boldsymbol{A} E_{1/p}(\boldsymbol{A}, \hat{t})$ *and* $\mathscr{D}^{1/p} \mathscr{E}_{1/p}(\boldsymbol{A}, \hat{t}) = \boldsymbol{A} \mathscr{E}_{1/p}(\boldsymbol{A}, \hat{t}) + \mathbb{I}\delta$.

(*b*) *If* $\boldsymbol{BC} = \boldsymbol{CA}$, *then,* $E_{1/p}(\boldsymbol{B}, t)\boldsymbol{C} = \boldsymbol{C} E_{1/p}(\boldsymbol{A}, t)$ *and* $\mathscr{E}_{1/p}(\boldsymbol{B}, t)\boldsymbol{C} = \boldsymbol{C}\mathscr{E}_{1/p}(\boldsymbol{A}, t)$.

Proof.

(i) The first part follows at once from the fact that $E_{1/p}(\boldsymbol{A}, 0) = \mathbb{I}$ and the ith column y_i of $E_{1/p}(\boldsymbol{A}, \hat{t})$ satisfies the following fractional differential equation: $\mathsf{d}^{1/p} y_i = \boldsymbol{A} y_i$ and $y_i(0) = e_i$, with e_i the ith unitary complex basis vector. The second is proven in Matignon & d’Adréa Novel (1995).

(ii) To prove the first part, note that

$$\boldsymbol{B}^k \boldsymbol{C} = \boldsymbol{C} \boldsymbol{A}^k,$$

whence

$$\begin{aligned} E_{1/p}(\boldsymbol{B}, \hat{t})\boldsymbol{C} &= \sum_{k=0}^{\infty} \frac{\boldsymbol{B}^k \hat{t}_+^{k/p}}{\Gamma(1 + k/p)} \boldsymbol{C}, \\ &= \sum_{k=0}^{\infty} \frac{(\boldsymbol{B}^k \boldsymbol{C}) \hat{t}_+^{k/p}}{\Gamma(1 + k/p)} \\ &= \sum_{k=0}^{\infty} \frac{(\boldsymbol{C} \boldsymbol{A}^k) \hat{t}_+^{k/p}}{\Gamma(1 + k/p)} \\ &= \boldsymbol{C} \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k \hat{t}_+^{k/p}}{\Gamma(1 + k/p)} \\ &= \boldsymbol{C}\boldsymbol{E}_{1/p}(\boldsymbol{A}, \hat{t}); \end{aligned}$$

the proof for $\mathscr{E}$ is analogous. □

The computation of the causal Mittag–Leffler matrix in the general case when the matrix A cannot be put in diagonal form is handled by considering each Jordan block separately. So, let $\lambda\mathbb{I}+\boldsymbol{\varepsilon}_1$, be the $r\times r$, $2\leqslant r\leqslant 8$, Jordan block associated to some eigenvalue λ, where $\boldsymbol{\varepsilon}_1$ is the $r\times r$ nilpotent matrix

$$\boldsymbol{\varepsilon}_1=\begin{bmatrix} 0 & 1 & 0 & \ldots 0 & \\ 0 & 0 & 1 & \ldots 0 & \\ 0 & 0 & 0 & \ldots 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots 1 & \\ 0 & 0 & 0 & \ldots 0 & \end{bmatrix}.$$

Then, since $\lambda\mathbb{I}$ and $\boldsymbol{\varepsilon}_1$ commute we have: (i) $\lambda\neq 0$

$$\begin{aligned}\mathscr{E}_{1/p}(\lambda\mathbb{I}+\boldsymbol{\varepsilon}_1,\hat{t}\,)&=\sum_{k=0}^{\infty}(\lambda\mathbb{I}+\boldsymbol{\varepsilon}_1)^k\frac{\hat{t}_+^{(1+k)/p-1}}{\Gamma[(1+k)/p]}\\ &=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\lambda^{k-i}\boldsymbol{\varepsilon}_1^i\frac{\hat{t}_+^{(1+k)/p-1}}{\Gamma[(1+k)/p]}\\ &=\sum_{k=0}^{r}\sum_{i=0}^{k}\lambda^{k-i}\boldsymbol{\varepsilon}_1^i\frac{\hat{t}_+^{(1+k)/p-1}}{\Gamma[(1+k)/p]}+\sum_{k=r+1}^{\infty}\sum_{i=0}^{r-1}\lambda^{k-i}\boldsymbol{\varepsilon}_1^i\frac{\hat{t}_+^{(1+k)/p-1}}{\Gamma[(1+k)/p]}\\ &=\mathscr{E}_{1/p}(\lambda,\hat{t}\,)\sum_{i=0}^{r-1}\lambda^{-i}\boldsymbol{\varepsilon}_1^i-\sum_{k=0}^{r-2}\sum_{i=k+1}^{r-1}\lambda^{k-i}\boldsymbol{\varepsilon}_1^i\frac{\hat{t}_+^{(1+k)/p-1}}{\Gamma[(1+k)/p]},\end{aligned}$$

where we have used the fact that $\boldsymbol{\varepsilon}_1^k=0$, for all $k\geqslant r$; and (ii) for $\lambda=0$

$$\begin{aligned}\mathscr{E}_{1/p}(\lambda\mathbb{I}+\boldsymbol{\varepsilon}_1,\hat{t})=\mathscr{E}_{1/p}(\boldsymbol{\varepsilon}_1,\hat{t})&=\sum_{k=0}^{\infty}\boldsymbol{\varepsilon}_1^k\frac{\hat{t}_+^{(1+k)/p-1}}{\Gamma[(1+k)/p]}\\ &=\sum_{k=0}^{r-1}\boldsymbol{\varepsilon}_1^k\frac{\hat{t}_+^{(1+k)/p-1}}{\Gamma[(1+k)/p]}.\end{aligned}$$

A.2. *Proof of Theorem 1*

Without any loss of generality, in this proof we consider the cylinder rotating counter-clockwise. The eigenvalues of the coefficient matrix $\boldsymbol{A}$ (compare (3.19)) are the roots of the characteristic polynomial $\mathsf{P}(\lambda;\hbar,Re_s)$. Since $\boldsymbol{A}$ is in its Frobenius block form, this can be readily calculated as

$$\begin{aligned}\mathsf{P}(\lambda;\,\hbar,Re_s)&=\det[\lambda^4+\boldsymbol{A}_{3/2}\lambda^3+\boldsymbol{A}_1\lambda^2+\boldsymbol{A}_{1/2}\lambda+\boldsymbol{A}_0]\\ &=\frac{Re_s^2(9+9\epsilon_L{}^2Re_s-6\epsilon_LRe_s{}^{3/2}+Re_s{}^2)}{729\hbar^2}-\frac{2Re_s{}^2(-3-3\epsilon_L{}^2Re_s+\epsilon_LRe_s{}^{3/2})}{81\hbar^{3/2}}\lambda\\ &\quad+\frac{Re_s{}^2(5+3\epsilon_L{}^2Re_s)}{27\hbar}\lambda^2+\frac{2Re_s{}^2}{9\sqrt{\hbar}}\lambda^3+\frac{(27\hbar+27\epsilon_L{}^2\hbar Re_s-6\epsilon_LRe_s{}^{3/2}+2Re_s{}^2)}{27\hbar}\lambda^4\\ &\quad+\frac{2(9\hbar+9\epsilon_L{}^2\hbar Re_s-\epsilon_LRe_s{}^{3/2})}{3\sqrt{\hbar}}\lambda^5+(2+9\hbar+9\epsilon_L{}^2\hbar Re_s)\lambda^6+6\sqrt{\hbar}\lambda^7+\lambda^8\end{aligned}$$

for all $\lambda\in\mathbb{C}$.

(*a*) and (*c*) The characteristic polynomial for $\hbar = 1/3$ can be factored as follows:

$$\mathsf{P}(\lambda; 1/3, Re_s) = \frac{1}{81}({Re_s}^2 + 9\lambda^4)[{Re_s}^2 - 6\epsilon_L {Re_s}^{3/2}(1+\sqrt{3}\lambda) \\ +9{\epsilon_L}^2 Re_s(1+2\sqrt{3}\lambda+3\lambda^2) + 9(1+2\sqrt{3}\lambda+5\lambda^2+2\sqrt{3}\lambda^3+\lambda^4)]$$

for all $\lambda \in \mathbb{C}$. So, $\mathsf{P}(\lambda; 1/3, Re_s)$ is an entire rational function of $\sqrt{Re_s}$ and λ and consequently defines an algebraic function $\lambda = \phi(\sqrt{Re_s})$. The critical points of this algebraic function are those points in $\mathbb{C}$ where $\mathsf{P}(\lambda; 1/3, Re_s)$ has multiple roots. Given the factorization of $\mathsf{P}(\lambda; 1/3, Re_s)$ shown above, it is enough to consider the critical points of the polynomial within brackets, which we denote by P_r, for the remaining factor provides no critical point for $Re_s \neq 0$.

Now, by definition, the critical points of P_r are those points where both the polynomial itself and its derivative P'_r are simultaneously null. To determine when this occurs we find it useful to compute the resultant $R(\mathsf{P}_r, \mathsf{P}'_r; 1/3, Re_s)$. Indeed, from Proposition 8.3 in Lang (1993), given two polynomials in $\mathbb{C}$, $p = a_0\hat{z}^n + \cdots + a_n$ and $\bar{p} = \bar{a}_0\hat{z}^m + \cdots + \bar{a}_m$ we have $R(p, \bar{p}) = a_0^m \bar{a}_0^n \Pi_{i=1}^n \Pi_{j=1}^m (o_i - \bar{o}_j)$, where o_i, $i = 1, \ldots, n$ and $\bar{o}_i$, $i = 1, \ldots, m$ are, respectively, the roots of p and $\bar{p}$, counted with multiplicities. Thus, $\mathsf{P}_r(\hat{z}; Re_s)$ possesses a multiple root if and only if $R(\mathsf{P}_r, \mathsf{P}'_r; Re_s) = 0$.

Recalling that the resultant $R(p, \bar{p})$ of two polynomials $p = a_0\hat{z}^n + \cdots + a_n$ and $\bar{p} = \bar{a}_0\hat{z}^m + \cdots + \bar{a}_m$ is defined (see Lang 1993) as the determinant

$$\underbrace{\begin{vmatrix} a_0 a_1 \ldots a_n \\ a_0 a_1 \ldots a_n \\ \cdots\cdots\cdots \\ a_0 a_1 \ldots a_n \\ \bar{a}_0 \bar{a}_1 \ldots \bar{a}_m \\ \bar{a}_0 \bar{a}_1 \ldots \bar{a}_m \\ \cdots\cdots\cdots \\ \bar{a}_0 \bar{a}_1 \ldots \bar{a}_m \end{vmatrix}}_{m+n},$$

where the blank space is filled with zeros, we obtain

$$R(\mathsf{P}_r, \mathsf{P}'_r) = 104\,976(\sqrt{Re_s})^4 (9{\epsilon_L}^2 + 3\epsilon_L\sqrt{Re_s} + (\sqrt{Re_s})^2)^2 \\ (9 + 90{\epsilon_L}^2(\sqrt{Re_s})^2 + 48\epsilon_L(\sqrt{Re_s})^3 + 16(\sqrt{Re_s})^4 + 81{\epsilon_L}^4(\sqrt{Re_s})^4).$$

From the previous expression, we conclude that the critical points of $\mathsf{P}(\lambda; 1/3, Re_s)$ are all non-real except for $Re_s = 0$ (just note that the coefficients in the factors of $R(\mathsf{P}_r, \mathsf{P}'_r)$ are all positive), where $\lambda = 0$ is a root of algebraic multiplicity four and $-\frac{1}{2}(\sqrt{3} \pm \mathrm{i})$ are the remaining roots, both of algebraic multiplicity two. Otherwise, $\mathsf{P}(\lambda; 1/3, Re_s) = 0$ with $Re_s > 0$ defines an eight-valued algebraic function $\lambda = \phi(Re_s)$ which can be explicitly determined (it is the product of two polynomials of

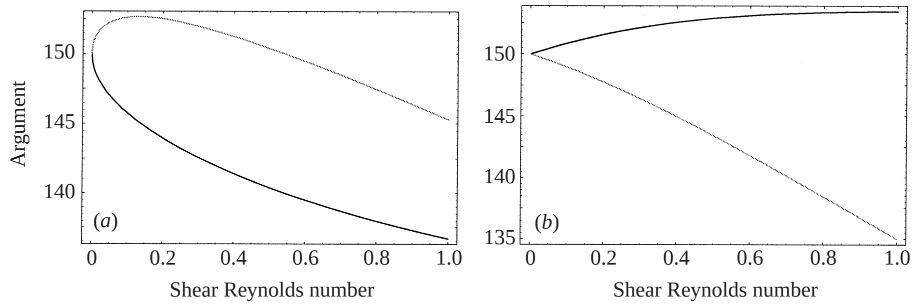

FIGURE 16. Argument of ϕ_6 (black line) and ϕ_7 (grey line) for $\hbar = 1/3$ as a function of Re_s: (a) with lift, (b) without lift.

fourth degree) as

$$\phi_{1,2} = \sqrt{\frac{Re_s}{3}}\mathrm{e}^{\pm \mathrm{i}\pi/4},$$

$$\phi_{3,4} = \sqrt{\frac{Re_s}{3}}\mathrm{e}^{\pm \mathrm{i}3\pi/4},$$

$$\phi_{5,6} = -\frac{3 \pm 3\mathrm{i}\epsilon_L\sqrt{Re_s} + \sqrt{-3 - 9{\epsilon_L}^2\, Re_s \pm \mathrm{i}(6\epsilon_L\sqrt{Re_s} + 4Re_s)}}{2\sqrt{3}},$$

$$\phi_{7,8} = -\frac{3 \pm 3\mathrm{i}\epsilon_L\sqrt{Re_s} - \sqrt{-3 - 9{\epsilon_L}^2 Re_s \pm \mathrm{i}(6\epsilon_L\sqrt{Re_s} + 4Re_s)}}{2\sqrt{3}}.$$

For $Re_s > 0$ the branches of this algebraic function are pairwise conjugate. Indeed, each of the pairs $\phi_{1,2}$ and $\phi_{3,4}$ are clearly conjugate and, for example, considering the pair $\phi_{5,6}$ we observe that the terms

$$3 \pm 3\mathrm{i}\epsilon_L\sqrt{Re_s}$$

are complex conjugates and the same can be said for the remaining terms inside the square root. Now, recall that the square root of two conjugate numbers are conjugate (it only halves the argument of both numbers), and so

$$\sqrt{-3 - 9{\epsilon_L}^2 Re_s \pm \mathrm{i}(6\epsilon_L\sqrt{Re_s} + 4Re_s)}$$

are complex conjugates.

The branches $\phi_{1,2}$ are the unique ones that lie in the boundary of $\Lambda_{\pi/4}$. Indeed, for small positive values of Re_s, which is the range of interest, we conclude by the Theorem on the continuity of the roots (see Knopp 1947) that $\arg(\lambda)$ is close to $\pm 5\pi/6$. In order to illustrate this point, in figure 16 the arguments of ϕ_6 and ϕ_7 are plotted as a function of Re_s for the range $Re_s \in [0, 1]$ and for $\epsilon_L = C_S/3\pi$ and $\epsilon_L = 0$. To conclude this part of the proof, we determine the variation of the branch $\phi_{1,2}$ as $\hbar$ changes. First, we note that for a fixed value of Re_s, $\mathsf{P}(\hat{\lambda}; \hbar, Re_s)\hbar^2$ is an entire rational function of $\sqrt{\hbar}$. So, analogously to the previous argument, it will define an algebraic function $\lambda = \bar{\phi}(\hbar)$. Moreover, there is a neighbourhood of $\hbar = 1/3$ where this algebraic function is regular. Now, by the Theorem on the differentiability of the roots (see Knopp 1947) there is a neighbourhood of $\hbar = 1/3$ where $\bar{\phi}_1$ is given by

$$\bar{\phi}_1(\hbar) = \bar{\phi}_1(1/3) + \bar{\phi}_1'(1/3)(\hbar - 1/3) + \mathsf{o}(\hbar - 1/3).$$

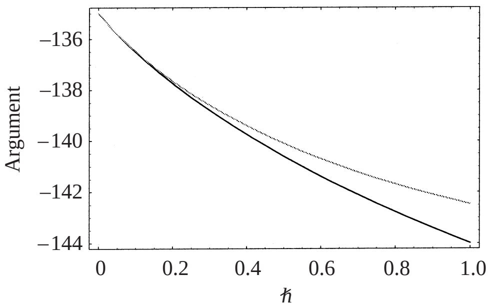

FIGURE 17. Argument of $\bar{\phi}_1'$ for $\hbar = 1/3$ as a function of Re_s: black line, with lift; grey line, without lift.

Hence, the variation of the root $\sqrt{Re_s/3}$ is determined by the derivative $\bar{\phi}_1'(1/3)$, which can be implicitly computed from the characteristic polynomial

$$\mathsf{dP}(\lambda(\hbar);\hbar, Re_s) = \frac{\partial \mathsf{P}}{\partial \lambda}(\lambda(1/3);1/3, Re_s)\frac{\mathrm{d}\lambda}{\mathrm{d}\hbar}(1/3) + \frac{\partial \mathsf{P}}{\partial \hbar}(\lambda(1/3);1/3, Re_s) = 0,$$

with $\lambda(1/3) = \sqrt{Re_s/3}$, yielding

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\hbar}(1/3)$$
$$= \frac{\sqrt{3}\sqrt{Re_s}(\sqrt{2}\sqrt{Re_s}(-6\mathrm{i} - 6\epsilon_L\sqrt{Re_s} + Re_s) - (1+\mathrm{i})(3+4\mathrm{i}Re_s + 3\epsilon_L\sqrt{Re_s}(-\mathrm{i}+Re_s)))}{(4+4\mathrm{i})(3-3\mathrm{i}\epsilon_L\sqrt{Re_s} + \mathrm{i}Re_s)\sqrt{Re_s} + 2\sqrt{2}(3+5\mathrm{i}Re_s + 3\epsilon_L\sqrt{Re_s}(-\mathrm{i}+Re_s))}.$$

Now, for small Re_s we expand in Taylor series to obtain

$$\arg\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\hbar}(1/3)\right) = -\frac{3\pi}{4} - \frac{Re_s}{3} + \mathsf{o}(Re_s),$$

which shows that for heavy (resp. light) particles we have $\hbar < 1/3$ (resp. $\hbar > 1/3$) and so $\arg(\bar{\phi}_1) < \Pi/4$ (resp. $\arg(\bar{\phi}_1) > \Pi/4$), as required. For illustration purposes, in figure 17 the argument of $\bar{\phi}_1'$ as a function of Re_s is shown in the range $Re_s \in [0, 1]$.

Note that although the lift has a fundamental influence on the equilibrium point, it produces little effect in the nature of the stability of the equilibrium point.

(*b*) The characteristic polynomial P has a pole at $\hbar = 0$. So, in order to study the stability of very heavy particles, it is convenient to undertake the following change of variables: $(\hbar, Re_s) \mapsto (\alpha = 2\hbar/(1-\hbar), \omega = Re_s/9\hbar)$. With respect to these new variables, the characteristic polynomial $\bar{\mathsf{P}}(\hat{\lambda};\alpha,\omega) = \mathsf{P}(\hat{\lambda};\hbar, Re_s)$ (extended by continuity at $\alpha = 0$) when computed in the limit of an infinitely heavy particle ($\alpha = 0$) is simply

$$\bar{\mathsf{P}}(\lambda;0,\omega) = \lambda^4 + 2\lambda^6 + \lambda^8 + \omega^2,$$

which can be solved by reducing the degree of the previous polynomial to four with the change of variable $\tilde{\lambda} = \lambda^2$:

$$\lambda_{1,2} = \pm\frac{\mathrm{i}\sqrt{2}}{2}\sqrt{1+\sqrt{1-4\mathrm{i}\omega}},$$

$$\lambda_{3,4} = \pm\frac{\sqrt{2}}{2}\sqrt{-1+\sqrt{1-4\mathrm{i}\omega}},$$

$$\lambda_{5,6} = \pm \frac{\mathrm{i}\sqrt{2}}{2}\sqrt{1+\sqrt{1+4\mathrm{i}\omega}},$$

$$\lambda_{7,8} = \pm \frac{\sqrt{2}}{2}\sqrt{-1+\sqrt{1+4\mathrm{i}\omega}}.$$

Realizing that $-1+\sqrt{1+4\mathrm{i}\omega}$ lies in the first quadrant for $\omega > 0$ we conclude the proof.

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